Math 23bTheoretical Linear Algebra and Multivariable Calculus II

PROBLEM SET 5

This Problem Set will be entirely devoted to complete the proof of the equivalence between the two definitions of compact sets, the one in terms of limit points and the one in terms of open coverings.

Let us first recall the two definitions of compactness. Let X be a metric space and let K be a subset of X.

Definition 0.1. K is said to be *compact* is for every infinite subset $S \subset K$ there exists $a \in K$ which is a limit point for S.

Definition 0.2. K is said to be *compact** if for every open cover of K there exists a finite subcover

In the last Problem Set we proved that compact* implies compact. In this Problem Set we'll prove the opposite direction:

Theorem 0.3. *If K is compact, then it is compact*.*

Problem 1: A set $A \subset X$ is said to be *dense* if $\bar{A} = X$. Prove that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Problem 2: A metric space X is called *separable* if it contains a countable dense subset. Prove that $\mathbb{R}^n$ is separable. (*Hint:* consider the set of all points with have only rational coordinates)

Problem 3: A collection $\{V_\alpha \mid \alpha \in I\}$ of open subsets of X is said to be a *base* for the topology of X if the following is true: for every $x \in X$ and every open set U which contains x, there is soma $\alpha \in I$ such that $x \in V_\alpha \subset U$. (Equivalently, every open set in X is the union of a subcollection of V_α's). Prove that every separable metric space has a countable base. (*Hint:* Take, as your base, all balls with rational radius and center in some countable dense subset of X)

Problem 4: Let K be a compact subset of X (according to Definition 0.1). Prove that K is separable. (*Hints: Step 1.* Fix $\delta > 0$ and pick $x_1 \in K$. By induction, having chosen $x_1, \dots, x_j \in K$,choose $x_{j+1} \in K$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for every $i = 1, \dots, j$. Prove that this process must stop after a finite number of steps. *Step 2.* Prove that K can be covered by finitely many balls of radius δ. *Step 3.* Take $\delta = 1, 1/2, 1/3, \dots$ and consider the set of all the centers of all the corresponding balls in Step 2. Prove that this set is countable and dense in K.)

Problem 5: Let K be a compact subset of X and let $\mathcal{C} = \{U_\alpha \mid \alpha \in I\}$ be an open cover of K. Prove that $\mathcal{C}$ admits a *countable* subcover $\Sigma = \{U_{\alpha_n} \mid n \in \mathbb{N}\} \subset \mathcal{C}$. (*Hints:* use the above results!)

Problem 6: Prove Theorem 0.3. (*Hint:* Let $\mathcal{C}$ be an open cover of K. By Problem 5, we can assume (without loss of generality) that $\mathcal{C} = \{U_1, U_2, \dots\}$ is a countable open over of K. Suppose, by contradiction, that no finite subset $\mathcal{C}_N = \{U_1, U_2, \dots, U_N\}$ is still an open cover of K. Let $F_N = K - (U_1 \cup U_2 \cup \cdots \cup U_N)$. By assumption each F_N is non-empty, but their intersection is empty. Let $S \subset K$ be an infinite set which contains a point from each F_N. Show that S does not have any limit point in K (thus contradicting the assumption that K is compact).)