Math 23bTheoretical Linear Algebra and Multivariable Calculus II

PROBLEM SET 6

Problem 1: Consider the function $\phi : \mathbb{R} \to \mathbb{R}^3$ given by

$$\phi(t) = (a\cos\omega t, a\sin\omega t, b\omega t) .$$

(a) The graph of this function defines a curve Φ in $\mathbb{R}^4$. Find the tangent line L to Φ at $t = 2$.

(b) Prove that there is no $\tau \in (0, 2\pi/\omega)$ such that $\phi(\frac{2\pi}{\omega}) - \phi(0) = \frac{2\pi}{\omega}\phi'(\tau)$. (**Note:** this shows, in particular, that the Mean Value Theorem does not hold for vector-valued functions.)

Problem 2: Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (\sin(x - y), \cos(x + y)) .$$

Find the tangent plane in $\mathbb{R}^4$ to the graph of f at the point $(\frac{\pi}{4}, \frac{\pi}{4}, 0, 0)$.

Problem 3: Given a set $D \subset \mathbb{R}^n$, a point $a \in D^\circ$, and a function $F : D \to \mathbb{R}^m$, we have seen in class how each of the following conditions implies the next one:

C1: $F(x)$ is continuously differentiable in an open set $U \subset D$ containing a,
C2: $F(x)$ is differentiable in an open set $U \subset D$ containing a,
C3: $F(x)$ is differentiable at $x = a$,
C4: $F(x)$ is continuous at $x = a$ and it has all directional derivatives at $x = a$,
C5: $F(x)$ has all directional derivatives at $x = a$,
C6: $F(x)$ has all partial derivatives at $x = a$.

In this Problem we'll see that none of the above statement implies any of the previous ones.

(a) Recall what each of the above statements means.

(b) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = \begin{cases} \frac{xy}{x-y} , & \text{for } x \neq y , \\ \frac{1}{x} , & \text{for } x = y \neq 0 , \\ 0 , & \text{for } x = y = 0 . \end{cases}$$

Prove that both partial derivatives of $f(x, y)$ are well defined at $(0, 0)$, but the directional derivative in the direction of $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not defined.
(Hence condition C6 holds, but none of the previous conditions C1–C5 holds)

(c) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4+y^2} , & \text{for } (x, y) \neq (0, 0) , \\ 0 , & \text{for } x = y = 0 . \end{cases}$$

Prove that all directional derivatives of $f(x, y)$ exist at $(0, 0)$, but f is not continuous at $(0, 0)$.

(Hence conditions C5 and C6 hold, but none of the previous conditions C1–C4 holds)

(d) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) \;=\; \begin{cases} \frac{x^2 y}{x^2+y^2} \;, & \text{for } (x,y) \neq (0,0) \;, \\ 0 \;, & \text{for } x = y = 0 \;. \end{cases}$$

Prove that all directional derivatives of $f(x,y)$ exist at $(0,0)$ and f is continuous at $(0,0)$, but f is not differentiable at $(0,0)$.
(Hence conditions C4–C6 hold, while conditions C1–C3 fail)

(e) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) \;=\; \begin{cases} 0 \;, & \text{for } x \neq \mathbb{Q} \;, \\ \frac{1}{q^3} \;, & \text{for } x = \frac{p}{q} \text{ in lowest terms} \;. \end{cases}$$

Prove that f is differentiable at $x = \sqrt{2}$ but f is not continuous (hence not differentiable) in any open interval containing $\sqrt{2}$.
(Hence conditions C3 holds, while C2 fails)

Hints:

1. Show that, for every rational $x = \frac{p}{q} \in \mathbb{Q}$ we have $|(\frac{p}{q})^2 - 2| \geq \frac{1}{q^2}$.
2. Deduce that $|\frac{p}{q} - \sqrt{2}| \geq \frac{1}{3q^2}$.
3. Deduce that $|f(\frac{p}{q}) - f(\sqrt{2})|/|\frac{p}{q} - \sqrt{2}| \leq \frac{3}{q}$.
4. Using the above facts, the problem is not very hard. You might find useful a result from a previous Problem Set.

(f) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) \;=\; \begin{cases} x^2 \sin\frac{1}{x} + y^2 \;, & \text{for } x \neq 0 \;, \\ y^2 \;, & \text{for } x = 0 \;. \end{cases}$$

Prove that f is differentiable at $(0,0)$ (and hence it is differentiable at every $(x,y) \in \mathbb{R}^2$), but the partial derivative $\frac{\partial f}{\partial x}$ is not continuous at $(0,0)$.
(Hence conditions C2–C6 hold, while condition C1 fails)