Math 23bTheoretical Linear Algebra and Multivariable Calculus II

PROBLEM SET 7

Problem 1: A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *homogeneous* if $f(tx) = tf(x)\,,\ \forall t \in \mathbb{R},\ x \in \mathbb{R}^n$.

(a) Prove that, if $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and homogeneous, then it must be such that $f(x) = f'(0)x\,,\ \forall x \in \mathbb{R}^n$.

(b) Prove that, if $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous, then all its directional derivatives at $0 \in \mathbb{R}^n$ exist, but $f'(0)$ does not exist unless f is a linear transformation.

(c) Prove that the functions $f, g : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) \;=\; \begin{cases} \frac{xy^2}{x^2+y^2} \;, & \text{for } (x,y) \neq (0,0) \\ 0 \;, & \text{for } (x,y) = (0,0) \end{cases}$$

and

$$g(x,y) \;=\; (x^{1/3} + y^{1/3})^3$$

have all directional derivatives at $(0,0)$ but they are not differentiable at $(0,0)$.

Problem 2: Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is in $C^2(\mathbb{R}^n)$. The square length of the *gradient* of f is:

$$||\nabla f||^2 \;=\; \left(\frac{\partial f}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial f}{\partial x_n}\right)^2 \;,$$

and the *Laplacian* of f is, by definition,

$$\nabla^2 f \;=\; \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} \;.$$

(They are both continuous functions in $\mathbb{R}^n$. I am omitting to write dependence on $(x_1, \ldots, x_n)$ in order not to simplify notation).

(a) $n = 2$, *polar coordinates.* Consider the function

$$g(r,\theta) \;=\; f(r\cos\theta, r\sin\theta) \;.$$

Use the chain rule to prove that

$$||\nabla f||^2 \;=\; \left(\frac{\partial g}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial g}{\partial \theta}\right)^2$$

and

$$\nabla^2 f \;=\; \frac{\partial^2 g}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} + \frac{1}{r}\frac{\partial g}{\partial r} \;.$$

(b) $n = 3$, *cylindrical coordinates.* Consider the function

$$g(r,\theta,z) \;=\; f(r\cos\theta, r\sin\theta, z) \;.$$

Use the chain rule to prove that

$$\nabla^2 f \;=\; \frac{\partial^2 g}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} + \frac{1}{r}\frac{\partial g}{\partial r} + \frac{\partial^2 g}{\partial z^2} \;.$$

(c) $n = 3$, *spherical coordinates.* Consider the function

$$\begin{aligned} h(\rho,\theta,\phi) &= f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \\ &= g(\rho\sin\phi, \theta, \rho\cos\phi)\ . \end{aligned}$$

Use the chain rule to prove that

$$\nabla^2 f = \frac{\partial^2 h}{\partial\rho^2} + \frac{2}{\rho}\frac{\partial h}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 h}{\partial\phi^2} + \frac{\cos\phi}{\rho^2\sin\phi}\frac{\partial h}{\partial\phi} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 h}{\partial\theta^2}\ .$$

Problem 3: (a) Find the Taylor series of $\sin x$ at $x = \pi/4$. Use it to compute the value of $\sin(50°)$ with accuracy of the first 4 decimal digits.

(b) Find the Taylor series of $\log x$ at $x = 1$. Use it to compute the value of $\log(\frac{3}{2})$ with accuracy of the first 3 decimal digits.

(c) Let $\alpha > 0$. Prove that the Taylor series of the function $(1+x)^\alpha$ at $x = 0$ is

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n\ ,$$

where $\binom{\alpha}{n} = \alpha(\alpha-1)\cdots(\alpha-n+1)/n!$. Prove that this Taylor series converges to $(1+x)^\alpha$ (i.e. this function is analytic) for $|x| < 1$.

Problem 4: This problem gives a form of "l'Hospital's rule". Suppose that f and g have k continuous derivatives in an open interval containing a, and suppose that both f and g and their first $k-1$ derivatives vanish at a. Prove that, if $g^{(k)}(a) \neq 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f^{(k)}(a)}{g^{(k)}(a)}\ .$$

(*Hint:* Use Taylor polynomials).