

# Math 23b Theoretical Linear Algebra and Multivariable Calculus II

## PROBLEM SET 8

**Problem 1:** Let  $f(x, y, z) = xy^2z^3$ , and consider the point  $a = (1, 0, -1)$ .

- (a) Find the second-order Taylor polynomial  $P_2(h)$  of  $f(x)$  at  $x = a$ .
- (b) Show directly (i.e. by computing explicitly the limit) that, as stated in the theorem proved in class, the second order remainder  $R_2(h) = f(a + h) - P_2(h)$  satisfies,

$$\lim_{h \rightarrow 0} \frac{R_2(h)}{\|h\|^2} = 0.$$

**Problem 2:** Find all critical points, and establish their nature (i.e. say whether they are local minima, local maxima, saddle points or none of them) for the following functions:

- (a)  $f(x, y) = 2x^2 - xy - 3y^2 - 3x + 7y$ ,
- (b)  $f(x, y) = x^2y^3(6 - x - y)$ ,
- (c)  $f(x, y) = \sin x \cosh y$ ,
- (d)  $f(x, y) = (x^2 + y^2)e^{x^2 - y^2}$ .

**Problem 3:** A wire of length 100 is cut into three pieces of lengths  $x$ ,  $y$  and  $100 - x - y$ . The first piece is bent into the shape of an equilateral triangle, the second is bent into the shape of a rectangle whose base is twice his height, and the third is made into a square. Find the minimum and the maximum of the sum of the areas. (Note: for obvious reasons, we require  $x, y, 100 - x - y \geq 0$ ).

**Problem 4:** Consider the following function:

$$f(x, y) = 3x^4 - 4x^2y + y^2.$$

- (a) Prove that the origin  $(0, 0)$  is a critical point.
- (b) Show that, on every line through the origin (i.e.  $y = mx$  or  $x = 0$ ), this function has a local minimum at the origin.
- (c) Surprisingly enough, the above fact does NOT imply that  $f(x, y)$  has a local minimum at the origin. In fact, prove that in every (arbitrarily small) disk around the origin, there are points  $(x, y)$  for which  $f(x, y) < f(0, 0)$ .

(Hint: in order to better understand what is going on, you can make a sketch indicating the set of points  $(x, y)$  at which  $f(x, y) > 0$  and the set of points  $(x, y)$  at which  $f(x, y) < 0$ .)

**Problem 5:** *Method of least squares.* Let  $x_1, \dots, x_n$  be  $n$  distinct real numbers, and suppose we are given  $n$  points in the plane with these numbers as first coordinates, i.e.  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ . In general it is impossible to find a straight line

$$y = f_{a,b}(x) = ax + b$$

which passes through all the points  $(x_i, y_i)$ ,  $i = 1, \dots, n$ . However, we would like to find the line which "best interpolates" all these points, namely which has minimal *total square error*:

$$E(a, b) = \sum_{i=1}^n [f_{a,b}(x_i) - y_i]^2 .$$

Determine the values of  $a$  and  $b$  which do this.

(*Note:* this method is used in all natural and social sciences, in all sort of purposes; if you haven't already done so, you will probably encounter it many times in your studies.)