# Math 23bTheoretical Linear Algebra and Multivariable Calculus II

PROBLEM SET 9

**Problem 1:** Use the method of Lagrange multipliers to find the points on the line $x + y = 10$ and the ellipse $x^2 + 2y^2 = 1$ which are closest.

**Problem 2:** Find the points in the closed unit ball

$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \le 1\}$$

where the function

$$f(x, y, z) = x^3 + y^3 + z^3$$

attains it maximum and minimum.

**Problem 3:** Given positive real numbers $x_1, \dots, x_n$, we define their arithmetic and geometric means as follows:

$$\begin{aligned} \text{AM} &= \frac{x_1 + \cdots + x_n}{n} \\ \text{GM} &= \sqrt[n]{x_1 \cdots x_n} \end{aligned}$$

Use the Lagrange multipliers to minimize the function

$$f(x_1, \dots, x_n) = \frac{x_1 + \cdots + x_n}{n}$$

on the set

$$S = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \cdots x_n = 1\} .$$

Use it to prove that the geometric mean is always less than or equal to the arithmetic mean.

**Problem 4:** In this problem you are supposed to go over the proofs we have seen in class and write them very carefully on your own, making sure that there are no gaps in the arguments. (Note: "it is obvious" is not an accepted argument!)

Recall that a *rectangle* $R$ in $\mathbb{R}^n$ is a set of type $R = I_1 \times \cdot \times I_n$, where $I_k$ are intervals in $\mathbb{R}$, a *polygon* $P$ is a finite union of rectangles, $P = R_1 \cup \cdots \cup R_N$, and a *partition* $\mathcal{P}$ of a polygon $P$ is a collection of rectangles $\mathcal{P} = \{R_1, \dots, R_N\}$ such that they are non-overlapping, $R_i^\circ \cap R_j^\circ = \emptyset$ for $i \ne j$, and their union is the whole polygon, $P = R_1 \cup \cdots \cup R_N$. We also define the *volume* of a rectangle $R = I_1 \times \cdot \times I_n$ to be $v(R) = (b_1 - a_1) \dots (b_n - a_n)$ if $I_k$ is an interval with extremes $a_k$ and $b_k$.

(a) Prove that if $R_1, \dots, R_N$ are arbitrary rectangles in $\mathbb{R}^n$, all contained in a bigger rectangle $R$, then there exists a partition $\mathcal{P} = \{P_1, \dots, P_k\}$ of $R$, consisting of disjoint rectangles, such that each of the $R_i$'s is union of some of the $P_j$'s.

(b) Prove that if $P = R_1 \cup \cdots \cup R_N$ is a polygon in $\mathbb{R}^n$, then you can write $P$ as a disjoint union of rectangles. Conclude, in particular, that every polygon admits a partition.

(c) Prove that if $\mathcal{P}_1$, $\mathcal{P}_2$ are partitions of the same rectangle $R$, then there exists a *refined* partition $\mathcal{P}$ of $R$, such that every element in $\mathcal{P}_1$ and $\mathcal{P}_2$ is a union of some of the elements of $\mathcal{P}$.

(d) Prove that if $R$ is a rectangle and $\mathcal{P} = \{R_1, \dots, R_N\}$ is any partition of $R$, then

$$v(R) \;=\; v(R_1) + \cdots + v(R_N) \ .$$

(e) Prove that if $\mathcal{P}$ is a polygon and $\mathcal{P}_1$, $\mathcal{P}_2$ are two partitions of $P$, then

$$\sum_{R \in \mathcal{P}_1} v(R) \;=\; \sum_{R \in \mathcal{P}_2} v(R) \ .$$

(Hence we can take this as our definition of volume $v(P)$ of the polygon $P$).

(f) Prove that if $P \subset Q$ are two polygons, then $v(P) \leq v(Q)$.

(g) Prove that if $P = R_1 \cup \cdots \cup R_N$ is a polygon, then $v(P) \leq v(R_1) + \cdots + v(R_N)$.