1 Problem Set 1 – Solutions

1.1 Problem 1

(1)   So you could do this problem in several ways. One thing that you shouldn't do is say 'Well, I proved it for n_0 and n_1 and a similar argument will show it for all n_i.' I didn't take off points for it because, in this case the argument is almost exactly the same for the general case as for the base case. But, writing this is a bad habbit to be in. In the future if you find yourself writing something like this you should use induction, or prove the general case instead of the base case (its frequently just as easy anyway). I will show the general case here.

Suppose $k \in \mathbb{N}$ is such that n_i is uniquely defined for all $i = 0, 1, 2, ...k-1$ (this is a fair assumption because k can be zero). We will show that k is unique. Suppose the contrary, that there exist n_k and m_k which satisfy

$$\sum_{i=0}^{k-1} \frac{n_i}{10^i} + \frac{n_k}{10^k} + \frac{n_{k+1}}{10^{k+1}} \le x < \sum_{i=0}^{k-1} \frac{n_i}{10^i} + \frac{n_k}{10^k} + \frac{n_{k+1} + 1}{10^{k+1}}$$

and

$$\sum_{i=0}^{k-1} \frac{m_i}{10^i} + \frac{m_k}{10^k} + \frac{m_{k+1}}{10^{k+1}} \le x < \sum_{i=0}^{k-1} \frac{m_i}{10^i} + \frac{m_k}{10^k} + \frac{m_{k+1} + 1}{10^{k+1}}$$

Now suppose, WLOG, that $m_k > n_k$. Then combining the first half of second inequality with the second half of the first one gives us

$$\sum_{i=0}^{k-1} \frac{m_i}{10^i} + \frac{m_k}{10^k} + \frac{m_{k+1}}{10^{k+1}} < \sum_{i=0}^{k-1} \frac{n_i}{10^i} + \frac{n_k}{10^k} + \frac{n_{k+1} + 1}{10^{k+1}}$$

But we chose k so that n_i is uniquely defined for all $i < k$, so $m_i = n_i$ for all $i < k$. Canceling and rearranging yields,

$$\frac{m_k - n_k}{10^k} < \frac{n_{k+1} - m_{k+1} + 1}{10^{k+1}}$$

But since, $m_k, n_k \in \mathbb{Z}$ with $m_k > n_k$, $m_k - n_k \ge 1$. Also, since $m_{k+1}, n_{k+1} \in \{0, 1, \ldots, 9\}$ (even when $k = 0$), $m_{k+1} - n_{k+1} + 1 \le 10$. So we get

$$\frac{1}{10^k} \le \frac{m_k - n_k}{10^k} < \frac{m_{k+1} - n_{k+1} + 1}{10^{k+1}} \le \frac{1}{10^k}$$

So $\frac{1}{10^k} < \frac{1}{10^k}$, a clear contradiction. Thus, n_i is uniquely defined for all $i \in \mathbb{N}$.

(2)   We must show that $S = \{x_0, x_1, \ldots\}$ is nonempty, and that it is bounded above. Clearly, S is bounded above by x as by definition, $x_k \leq x$ for all k. Also, S is nonempty as $x_0 = n_0 \in S$ where n_0 is the greatest integer not exceeding x, which exists by problem 2, part 2.

NB: Many of you did not prove the existence of x_0, but since most of you got problem 2, part 2 correct, I wasn't a stickler about this.

(3)   To show that anything is a least upper bound requires two things. First, that it is an upper bound, and second, that no value less than it is an upper bound. We have already shown that x is an upper bound for S in part 2. So it remains to show that for any $\epsilon > 0$, $x - \epsilon$ is not an upper bound. So choose $\epsilon > 0$.

Lemma For every $\alpha \in \mathbb{R}$, there exists a $k \in \mathbb{N}$ such that $10^k > \alpha$.

Proof Suppose that for some $\alpha \in \mathbb{R}$ no such $k \in \mathbb{N}$ exsisted. Then define $S = \{10^k : k \in \mathbb{N}\}$. Clearly, $S \subset \mathbb{R}$ and by assumption, α is an upper bound of S. Then by the least upper bound axiom, S has a least upper bound. Let $\beta = supS$. Then by definition, $\frac{\beta}{10}$ is not an upper bound of S. So there exists some $k \in \mathbb{N}$ such that $\frac{\beta}{10} < 10^k \Rightarrow \beta < 10^{k+1}$. But $10^{k+1} \in S$, so β is not an upper bound of S, which contradicts our original assumption. Therefore, S is unbounded.

So now, calculate the real number $\frac{1}{\epsilon}$. By our lemma, there exists a $k \in \mathbb{N}$ such that $10^k > \frac{1}{\epsilon} \Rightarrow \frac{1}{10^k} < \epsilon$. So suppose that $x - \epsilon$ is an upper bound. Then $x - \frac{1}{10^k}$ is an upper bound also as, $x - \epsilon < x - \frac{1}{10^k}$. So then $x_k < x - \frac{1}{10^k} \Rightarrow n_0 + \frac{n_1}{10} + \cdots + \frac{n_k+1}{10^k} < x$ which contradicts the definition of n_k. Therefore, $x - \epsilon$ is not an upper bound for S, for all $\epsilon > 0$. And so $supS = x$, as desired.

1.2   Problem 2

(1)   We will use the Archimedian property twice. The Archimedian property states for any $x, y \in \mathbb{R}$ where $y > 0$ we may find a natural number $n \in \mathbb{N}$ such that $x < ny$. The first time, we let $y = 1$ to find $n \in \mathbb{N}$ such that $n > x$, and the second time we let $y = 1, x = -x$ to find $a \in \mathbb{N}$ such that $a > -x \Rightarrow -a < x$. So if we let $m = -a$, we have found $m, n \in \mathbb{Z}$ such that $m < x < n$, as desired.

(2)   From part 1, we know that for any real x we may find $m, n \in \mathbb{Z}$ such that $m < x < n$. So for some $x \in \mathbb{R}$ we define $S = \{a \in \mathbb{Z} : m \le a \le x\}$. Then, clearly $S \subset \mathbb{R}$, and x is an upper bound for S, so S has a least upper bound. But S is finite. So S has a greatest element. Let this element be n. Then, clearly, $n + 1 \notin S$, since otherwise, n would not be the greatest element of S. But $n \in S \Rightarrow n \ge m \Rightarrow n + 1 > m$. So for $n + 1$ to not be in S it must be that $n + 1 > x$. Also, since $n \in S$, we see that $n \le x$. This gives us $n \le x < n + 1$. So we have shown that an n must exist. To show that it is unique, simply suppose that two such integers, say m and n, satisfy the desired inequality, and WLOG suppose that $n < m$. But $n, m \in \mathbb{Z} \Rightarrow n + 1 \le m$. This gives us $n \le x < n + 1 \le m \le x < m + 1 \Rightarrow x < x$, an obvious contradiction. Therefore, for each $x \in \mathbb{R}$ we may find an $n \in \mathbb{N}$ such that $n \le x < n + 1$, and this n is unique.

Also, notice that for each $x \in \mathbb{R}$ there exists a unique $y \in \mathbb{R}$ such that $-x = y$. This gives us $-n-1 < y \le -n$, and so if we let $m = -n-1$ we have that for each real y there exists a unique $m \in \mathbb{Z}$ such that $m < y \le m + 1$.

(3)   From part 2, we know that for every $x \in \mathbb{R}$ there is a unique $n \in \mathbb{Z}$ such that $n < x \le n + 1$. So $n < x \Rightarrow n + 1 < x + 1$, which gives us $x \le n+1 < x+1$, so if we let $m = n+1$ then we have $x \le m < x + 1$. Also, since n was unique, we know that m is unique.

(4)   The Archimedian property of the real numbers tells us that for any $y \in \mathbb{R}$ we may find an $n \in \mathbb{N}$ such that $n > y$. If $y > 0$ then there exists a unique $x > 0$ such that $y = \frac{1}{x}$. So then we get $n > \frac{1}{x} \Rightarrow \frac{1}{n} < x$.

(5)   Recall from class that for any two distinct real numbers $x, y \in \mathbb{R}$ such that $y > x$ there exists a rational number r such that $x < r < y$. But then, since $r, y \in \mathbb{R}$ with $y > r$, we may find rational number r_1 such that $x < r < r_1 < y$. We repeat this argument inductively so that if we are given n rational numbers between x and y such that $x < r < x_1 \cdots < r_{n-1} < y$, we may find $r_n \in \mathbb{Q}$ such that $x < r < x_1 \cdots < r_n < y$. Therefore, by induction, we may find infinitely many distinct rational numbers between x and y.

(6)   Choose two distinct real numbers x and y, and say WLOG that $y > x$. Clearly their mean, $\frac{x+y}{2}$ will lie between them. So if $\frac{x+y}{2}$ is irrational,

then we are done. So suppose that it is rational. Now calculate the difference between the mean of x and y and y, and call it ϵ. So $y - \frac{x+y}{2} = \frac{y-x}{2} = \epsilon$. We will show that there is an irrational number within ϵ of $\frac{x+y}{2}$. As we will show in part 9, since $\frac{x+y}{2} \in \mathbb{Q}$, if $\alpha \notin \mathbb{Q}$, then $\frac{x+y}{2} + \alpha \notin \mathbb{Q}$. Therefore, if we can find a positive irrational number, α which is less than ϵ then we are done, as $\frac{x+y}{2} + \alpha < y$, and so we will have found an irrational number inside the interval. To show that there are infinitely many irrational numbers we just use induction exactly as in part 5. We now show that for every $\epsilon > 0$ there exists a positive irrational number α such that $\alpha < \epsilon$. Recall from last semister that $\sqrt{2}$ is irrational. As we will show in part 9, that means that $\frac{\sqrt{2}}{n}$ is also irrational for all nonzero integers n. We now use the Archimedian property to see that for any $\epsilon > 0$ there exists an $n \in \mathbb{N}$ such that $\sqrt{2} < n\epsilon \Rightarrow \frac{\sqrt{2}}{n} < \epsilon$. Then $\alpha = \frac{\sqrt{2}}{n}$ is our irrational number which is less than ϵ, and so we are done.

(7) If $x, y \in \mathbb{Q}$ then so are $x + y, x - y, xy, \frac{x}{y}$. This follows from the field axioms of Q. We now give explicit calculations of each, given that $x = \frac{m}{n}$, $y = \frac{p}{q}$. We get that $x + y = \frac{mq+np}{nq}$, $x - y = \frac{mq-np}{np}$, $xy = \frac{mp}{nq}$, $\frac{x}{y} = \frac{mq}{np}$. Because addition and multiplication is closed over the integers (meaning the sum and product of two integers are integers), it is clear that each of these values are rational.

Note I am going to go out of order and do 9 before 8 because I will use the result of 9 in 8 several times.

(9) If $x \in \mathbb{Q}$ and $y \in \mathbb{R} - \mathbb{Q}$ then $x + y, x - y \in \mathbb{R} - \mathbb{Q}$. Let $x + y = z$. Then if z is rational, we have $y = z - x$. But by part 7, $z - x \in \mathbb{Q}$ since it is the difference of two rationals. So we have a rational equal to a rational. Similarly, let $x - y = z$ and suppose $x \in \mathbb{Q}$. Then we get $y = x - z \Rightarrow y \in \mathbb{Q}$, again by part 7. But y is assumed to be irrational, so we have a contradiction. Now, we examine xy and $\frac{x}{y}$. Cearly, in either case, if $x = 0$ then so too $xy = \frac{x}{y} = 0$. So we have at least one case where $xy, \frac{x}{y} \in \mathbb{Q}$. Now suppose $x \neq 0$. So let $xy = z$. If $z \in \mathbb{Q}$, then we get $y = \frac{z}{x} \in \mathbb{Q}$ by part 7. Similarly, if $\frac{x}{y} = z$, and $z \in \mathbb{Q}$ then we get $y = \frac{x}{z} \in \mathbb{Q}$ by part 7. Clearly, each is a contradiction as $y \in \mathbb{R} - \mathbb{Q}$. So we see that unless $x = 0$ (in which case the product and quotient are zero, and therefore rational), the sum, difference, product and quotient are irrational.

(8)   Now suppose that $x, y \in \mathbb{R} - \mathbb{Q}$. Then we will show by example that we may make no statement about any of $x + y, x - y, xy, \frac{x}{y}$. For our examples, we will use the fact that $\pm\sqrt{2}, \sqrt{3}, \sqrt{\frac{2}{3}}, \sqrt{6} \in \mathbb{R} - \mathbb{Q}$. All of these follow, either from part 8, or from the fact, discussed last semister, that the squareroot of any nonperfect square is irrational. Suppose $x = y = \sqrt{2}$. Then $x + y = 2\sqrt{2} \in \mathbb{R} - \mathbb{Q}$, $x - y = 0 \in \mathbb{Q}$, $xy = 2 \in \mathbb{Q}$, and $\frac{x}{y} = 1 \in \mathbb{Q}$. Now suppose, $x = \sqrt{2}$, $y = -\sqrt{2}$. Then $x + y = 0 \in \mathbb{Q}$ and $x - y = 2\sqrt{2} \in \mathbb{R} - \mathbb{Q}$. Now, suppose that $x = \sqrt{2}$, $y = \sqrt{3}$. Then $xy = \sqrt{6} \in \mathbb{R} - \mathbb{Q}$, and $\frac{x}{y} = \sqrt{\frac{2}{3}} \in \mathbb{R} - \mathbb{Q}$. So we have constructed examples which show that no statement can be made about the rationality of $x + y, x - y, xy, \frac{x}{y}$ when $x, y \in \mathbb{R} - \mathbb{Q}$.

1.3 Problem 3

We must show that every $n \in \mathbb{N}$ is either even or odd, and that no integer can be both. So first, we show by induction that every integer is either even or odd. Clearly, $1 = 2 * (0) + 1$ is odd (since 0=2*0 is even). Now, suppose that all of the integers less than n are either even or odd. If $n - 1$ is even then by definition, n is odd since $n = (n - 1) + 1$. On the other hand, if $n - 1$ is odd, then by definition $n - 2$ is even. So $n - 2 = 2k \Rightarrow n = 2(k + 1)$, and so n is even. Therefore, every positive natural number is either even or odd. Now, suppose that some n is both even and odd. Then because n is odd, $n - 1$ must be even. So there exist natural numbers $p, q \in \mathbb{N}$ such that $n - 1 = 2p$, $n = 2q$. So then $2p + 1 = 2q \Rightarrow 1 = 2(q - p)$. But $q - p \in \mathbb{Z}$, so clearly $q - p > 0$ since $q - p \leq 0 \Rightarrow 2(q - p) \leq 0 \Rightarrow 2(q - p) \neq 1$. But $q - p > 0 \Rightarrow q - p \geq 1 \Rightarrow 2(q - p) \geq 2 \Rightarrow 2(q - p) \neq 1$. So we see that there can be no $n \in \mathbb{N}$ such that n is both even and odd.

1.4 Problem 4

(1)   If we let $\alpha = supA$ then clearly $-\alpha$ is a lower bound for B since suppose that there exists a $b \in B$ with $b < -\alpha$. But $b \in B \Rightarrow b = -a$ for some $a \in A$. So $-a < -\alpha \Rightarrow a > \alpha$ which contradicts the assumption that α is an upper bound for A. Therefore, B is bounded below. Now, we will show that there for every $\epsilon > 0$, $-\alpha + \epsilon$ is not a lower bound for B. Recall that $\alpha = supA$ which means that for any $\epsilon > 0$, $\alpha - \epsilon$ is not an upper bound, which means that for any $\epsilon > 0$ there exists an $a \in A$ such that $a > \alpha - \epsilon \Rightarrow -a < -\alpha + \epsilon$, but $-a \in B$. Therefore, $-\alpha = infB$. So $infB = -supA$ as desired.

(2)   By definition, $supA$ is defined so that for all $a \in A$, $a \leq supA$. Similarly, for all $b \in B$, $b \leq supB$. Therefore, for any $c \in C$, by definition $c = a + b \leq supA + supB$. So $supA + supB$ is an upper bound of C. Now, we will show that for all $\epsilon > 0$, $supA + supB - \epsilon$ is not an upper bound for C. We start by choosing $\epsilon > 0$, and calculating $\frac{\epsilon}{2}$. By definition of least upper bound, there exists some $a \in A$ such that $a > supA - \frac{\epsilon}{2}$ and likewise, there exists a $b \in B$, such that $b > supB - \frac{\epsilon}{2}$. Therefore, choosing these a and b we get a $c = a + b > supA - \frac{\epsilon}{2} + supB - \frac{\epsilon}{2} = supA + supB - \epsilon$. So for any $\epsilon > 0$ we may find a $c \in C$ so that $c > supA + supB - \epsilon$. Therefore, $supC = supA + supB$ as desired.