Problem Set 2 Solutions

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1 Problem 1

Need to show 3 things for equivalence "concepts":

(i) reflexivity, $A \approx A$: Obvious bijection is the identity function:

$$\phi : A \to A, \quad \phi(a) = a \quad \forall a \in A.$$

This is clearly bijective, and $\phi^{-1} = \phi$.

(ii) symmetry, $A \approx B \Rightarrow B \approx A$: Since $A \approx B$, $\exists$ bijective map $\phi : A \to B$. Define $f : B \to A$ as $f(b) = \phi^{-1}(b)$, which exists since ϕ is bijective. Then f is also bijective since it has an inverse, namely $f^{-1} = \phi$.

(iii) transitivity, $A \approx B, B \approx C \Rightarrow A \approx C$: Let $\phi : A \to B$ and $\chi : B \to C$ be bijective maps, and define $f : A \to C, \quad f(a) = (\chi \circ \phi)(a)$. The invertibility of f follows from the invertibility of ϕ and χ as follows. Define the function $g : C \to A$ as $g(c) = (\phi^{-1} \circ \chi^{-1})(c)$. To show that g is the inverse of f, suffices to show that $f \circ g = g \circ f =$identity:

$f \circ g = (\chi \circ \phi) \circ (\phi^{-1} \circ \chi^{-1}) = \chi \circ (\phi \circ \phi^{-1}) \circ \chi^{-1} = \chi \circ \chi^{-1}$= identity, and similarly $g \circ f = (\phi^{-1} \circ \chi^{-1}) \circ (\chi \circ \phi) = \phi^{-1} \circ (\chi^{-1} \circ \chi) \circ \phi = \phi^{-1} \circ \phi$= identity.

2 Problem 2

Note: There are two working definitions of the word 'countable,' one that includes finite sets as countable and one that doesn't. Both definitions are acceptable as long as you specify which one you're using and you're consistent and complete. Here I use the latter definition and include a note at the end about the former.

(1) Suppose $\{S_i\}$ is a countable family of countable sets. Since $S_1 \subseteq \cup_i S_i$, it is clear that the cardinality of the union $C(\cup_i S_i)$ is at least countable. Also, $C(\cup_i S_i) \leqslant \sum_i C(S_i)$ since some of the elements may exist in more than one of the S_i's, so it suffices to show that the larger of the two, $\sum_i C(S_i)$, is at most countable. This set is easier to work with because we just count all the repetitions and so don't have to worry about "skipping" elements when we count. We begin by arranging the elements in an array as follows.

	1	2	3	$\cdots$
S_1	s_{11}	s_{12}	s_{13}	$\cdots$
S_2	s_{21}	s_{22}	s_{23}	$\cdots$
S_3	s_{31}	s_{32}	s_{33}	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

From this it is clear that there is a bijection between the elements of this array and $\mathbb{N} \times \mathbb{N}$, which we know from class is countable. Thus we have shown that a countable union of countable sets is countable.

If we include finite in the definition of countable, then there are 3 additional cases:

i. a finite union of finite sets is finite (and countable).

ii. a finite union of infinite and countable sets is infinite and countable

iii. an infinite and countable union of finite sets may be finite or infinite (but still countable), depending on how much they overlap.

(2) This problem can be done in 3 steps. In i. show that for a given positive integer N, the number of $(k+1)$-tuples of integers which satisfy $k+|n_0|+|n_1|+\cdots+|n_k| \leqslant N$ is finite. In ii. argue that this provides a way of enumerating all the polynomials of finite degree with integral coefficients, showing that they are countable. In iii. show that the algebraic numbers, which are the solutions to these polynomials, are also countable.

i. Each of the n_i must lie in the set $\{-N, -N+1, \cdots, N-1, N\}$ which has cardinality $2N+1$, and so the number of $(k+1)$-tuples is at most $(k+1)(2N+1)$, which is finite.

ii. Consider the set of all the tuples described in part i, for all N. This set is countable, as a countable union of finite sets. There is a simple bijection between this set and the set of polynomials with integral coefficients, namely

$(n_0, n_1, \cdots, n_k) \mapsto n_o x^k + n_1 x^{k-1} + \cdots + n_{k-1} x + n_k$.

Thus the set of these polynomials is also countable.

iii. Each polynomial of degree k has at most k real roots. In particular, the set consisting of the roots of a polynomial with integral coefficients is finite. Thus the union of all such sets, which is the set of algebraic numbers, is countable, as the countable union of finite sets.

3 Problem 3

(1) Proof by contradiction (Cantor's diagonal argument):

Suppose the set $\{0,1\}^{\mathbb{N}} = \{(x_1, x_2, \cdots) | x_k = 0 \text{ or } 1\}$ is countable. Then the elements of this set can be listed in order, and we can set up an array similar to the one before:

$$\begin{matrix} (x_{11}, & x_{12}, & x_{13}, & \cdots) \\ (x_{21}, & x_{22}, & x_{23}, & \cdots) \\ (x_{31}, & x_{32}, & x_{33}, & \cdots) \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

Now we construct a sequence as follows. Take the sequence of diagonal entries $\{x_{ii}\}$ and switch every 1 to 0 and 0 to 1. This new sequence differs from every sequence listed above in at least one element, demonstrating that no such list can include all elements of $\{0,1\}^{\mathbb{N}}$. This proves the set is uncountable.

(2) Need to find a bijection $\phi : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$. Namely, we need a way to make 2 such sequences from one, such that we can reverse the process and recover the initial sequence. For example, the map $\phi(x_1, x_2, \cdots) = ((x_1, x_3, \cdots), (x_2, x_4, \cdots))$ works. It is clearly a bijection, but for the sake of rigor we write down its inverse:

$\phi^{-1}((x_1, x_2, \cdots), (y_1, y_2, \cdots)) = (x_1, y_1, x_2, y_2, \cdots)$.

4 Problem 4

To prove that the set $C[a,b] = \{f : [a,b] \to \mathbb{R} | f \text{ continuous}\}$ endowed with the metric $d(f,g) = sup\{|f(x) - g(x)||a \leqslant x \leqslant b\}$ is a metric space, we need to show 3 things.

(i) symmetry: $d(f,g) = sup\{|f(x) - g(x)|\} = sup\{|g(x) - f(x)|\} = d(g,f)$

(ii) positive definiteness: $d(f,g) = sup\{|f(x) - g(x)|\} \geqslant 0$.

If $d(f,g) = 0 = sup\{|f(x) - g(x)||a \leqslant x \leqslant b\}$, then $f = g$ since if the supremum of a non-negative set is 0, every element must be 0. Also, it is clear if $f = g$ then $d(f,g) = 0$.

(iii) triangle inequality:

$$\begin{aligned} d(f,h) &= sup\{|f(x) - h(x)|\} \\ &= sup\{|f(x) - g(x) + g(x) - h(x)|\} \\ &\leqslant sup\{|f(x) - g(x)| + |g(x) - h(x)|\} \\ &\leqslant sup\{|f(x) - g(x)|\} + sup\{g(x) - h(x)|\} \\ &= d(f,g) + d(g,h) \end{aligned}$$

5 Problem 5

The discrete metric is $d(x,y) = 1$ if $x \neq y$ and $= 0$ if $x = y$. I find it is easier to first figure out what open and closed sets look like (5, 6) and then answer the first 4 parts:

(5) Each point in $\mathbb{R}^2$ is an open set, since the open ball $S(a, \frac{1}{2})$ contains only the point a. (That an open ball is an open set is proved in problem 6). It follows that every set M is open, as a union of open sets (namely all the points in M.)

(6) This in turn implies that every set is also closed, since its complement is open.

(1) a is a contact point of $M \Leftrightarrow a \in M$. As a closed set, M contains all of its contact points.

(2) M has no limit points: For every point $a \in \mathbb{R}^2$ there exists an open set which does not contain any points of M other than perhaps a, namely the set $\{a\}$.

(3) a is an interior point of $M \Leftrightarrow a \in M$. Since M is open, every point in M is an interior point.

(4) a is an isolated point of M if there exists an open set containing a and no other point of M. Such an open set exists for every a in M, namely $\{a\}$. Thus every point of M is an isolated point. All points are isolated

6 Problem 6

(1) To prove that the open ball $S(a, r)$ is an open set in a metric space, need to show:

$\forall b \in S(a, r), \; \exists \, \epsilon > 0$ such that $S(b, \epsilon) \subset S(a, r)$.

For this we employ the triangle inequality. Suppose that $d(a, b) = x$, and choose $\epsilon < r - x$. Let $c \in S(b, \epsilon)$, namely $d(b, c) < \epsilon$. Then

$$\begin{aligned} d(a, c) &\leqslant d(a, b) + d(b, c) \\ &\leqslant x + \epsilon \\ &\leqslant x + (r - x) \\ &= r. \end{aligned}$$

Thus $c \in S(a, r) \; \forall \; c \in S(b, \epsilon)$, namely $S(b, \epsilon) \subset S(a, r)$.

(2) To prove that the closed ball $S[a, r]$ is a closed set, we show that its complement is open, namely:

$\forall b \notin S[a, r], \; \exists \, \epsilon > 0$ such that $S(b, \epsilon) \bigcap S[a, r] = \emptyset$.

Since $b \notin S[a, r]$, $d(a, b) = x > r$. Let $\epsilon < x - r$, and let c be a point in $S(b, \epsilon)$. Rearranging $d(a, b) \leqslant d(a, c) + d(b, c)$, can write

$$d(a, c) \geqslant d(a, b) - d(b, c)$$

$$
\begin{aligned}
&> x - \epsilon \\
&> x - (x - r) \\
&= r
\end{aligned}
$$

Thus $c \notin S[a, r] \ \forall \ c \in S(b, \epsilon)$, namely $S(b, \epsilon) \bigcap S[a, r]$ is indeed $\emptyset$.

(3) If $M_1 \subset M_2$, want to show that $\bar{M}_1 \subseteq \bar{M}_2$. Let a be a point in $\bar{M}_1$. There are two possibilites:

If $a \in M_1$ then $a \in M_2 \subset \bar{M}_2$ and we're done.

If $a \notin M_1$ then a is a limit point of M_1, which means that $\forall r > 0, \ S(a, r)$ contains at least one point, call it b, which is in M_1. But $b \in M_1 \subset M_2$ so every $S(a, r)$ also contains at least one point in M_2, making a a limit point of M_2. Thus $a \in \bar{M}_2$, which completes the proof.