Problem Set 3 Solutions
Math 23b

1) Let $(x_1, x_2, \ldots)$ be a sequence in a metric space X. Prove or disprove the following statements:

(1) if there is a unique limit point x for the sequence, then the sequence converges to x.

False. Counterexample: Consider the sequence $(0, 1, 0, 2, 0, 3, 0, 4, \ldots)$. 0 is the only limit point of the sequence, but the sequence does not converge to 0.

(2) if x is a limit point of the set $\{x_n \mid n \in \mathbb{N}\}$, then x is a limit point for the sequence $(x_1, x_2, \ldots)$.

True. Proof: Assume x is a limit point of $\{x_n \mid n \in \mathbb{N}\}$. Then, by definition, for each $r > 0$, there are an infinite number of $x_n \neq x$ such that $\mid x - x_n \mid < r$. So, for each $m \in \mathbb{N}$, we can pick $y_m \in \{x_n\}$ such that $\mid y_m - x\} < 1/2^m$ and y_m is different from $y_1, y_2, \ldots, y_{m-1}$. Then, the sequence $(y_1, y_2, \ldots)$ has x as a limit point.

(3) if x is a limit point of the sequence $\{x_n \mid n \in \mathbb{N}\}$ then x is a limit point for the sequence $\{x_1, x_2, \ldots\}$.

False. Counterexample: The sequence $\{0, 0, \ldots\}$ has 0 as a limit point but the set $\{x_n\}$ has only the element 0, so it has no limit points.

2) Prove the following:

The sequence $(x_1, x_2, \ldots)$ converges to x if and only if for every open set $U \subset X$ such that $x \in U$ we have $x_n \in U$ for all but finitely many n's.

Assume $(x_1, x_2, \ldots)$ converges to x. Let U be an open set containing x. Then, since U is open, there exists some $r > 0$ such that $S(x, r) \subset U$. However, by the definition of convergent sequences, we can find $N \in \mathbb{N}$ such that for all $n > N$, $x_n \in S(x, r)$. Since $S(x, r) \subset U$, $x_n \in U$. Therefore, at most N of the x_n are not in U, so only a finite number of x_n fall outside of any open U containing x.

For the converse, assume that for every open set U containing x, only a finite number of x_n are not in U. Then, for any $r > 0$, only a finite number of x_n fall outside of $S(x, r)$. However, since this is a finite subset of the x_n, there must be one element with a maximal n which we shall denote as N. Then, for each $r > 0$, we can find $N \in \mathbb{N}$ such that if $n > N$, $x_n \in S(x, r)$. This is the definition of convergence of a sequence, so $(x_1, x_2, \ldots)$ converges to x.

3) (1) Show that if $(x_1, x_2, \ldots)$ converges to x and $(y_1, y_2, \ldots)$ converges to y, then $(x_1y_1, x_2y_2, \ldots)$ converges to xy.

Choose $\epsilon > 0$. We wish to show that we can find N such that for all $n > N$, $\mid x_ny_n - xy \mid < \epsilon$. We can rewrite:

$$\mid x_ny_n - xy \mid = \mid (x_n - x + x)(y_n - y + y) - xy) \mid = \mid (x(y_n - y) + y(x_n - x) + (x_n - x)(y_n - y) \mid$$

By the triangular inequality, we can break this up:

$$\mid x(y_n - y) + y(x_n - x) + (x_n - x)(y_n - y) \mid \leq \mid x(y_n - y) \mid + \mid y(x_n - x) \mid + \mid (x_n - x)(y_n - y) \mid$$

So, if we can find N such that for $n > N$, we have $\mid x(y_n - y) \mid < \epsilon/3$, $\mid y(x_n - x) \mid < \epsilon/3$, and $\mid (x_n - x)(y_n - y) \mid < \epsilon/3$, we are done. However, if $y \neq 0$, we can find N_1 such that $\mid x - n_x \mid < \epsilon/(3y)$ for $n > N_1$ since $(x_1, x_2, \ldots)$ converges to x.; if $y = 0$, then we can just let $N_1 = 0$. Likewise, if $x \neq 0$, we can find N_2 such that $\mid y - n_y \mid < \epsilon/(3x)$ for $n > N_2$; if $x = 0$, then we can just let $N_2 = 0$. We can also find N_3 such that $\mid y_n - y \mid < \sqrt{\epsilon}$ for $n > N_3$ and N_4 such that $\mid x_n - x \mid < \sqrt{\epsilon}$ for $n > N_4$. Then, let $N = max\{N_1, N_2, N_3, N_4\}$. For all $n > N$, we clearly have $\mid x(y_n - y) \mid < \epsilon/3$, $\mid y(x_n - x) \mid < \epsilon/3$, and $\mid (x_n - x)(y_n - y) \mid < \epsilon/3$, so $(x_1y_1, x_2y_2, \ldots)$ converges to xy.

(2) If $x_1, x_2, \ldots$ and x are all non-zero, prove that $(\frac{1}{x_1}, \frac{1}{x_2}, \ldots$ converges to $\frac{1}{x}$.

Choose $\epsilon > 0$. By the definition of convergence can find N_0 such that $\mid x_n - x \mid < \epsilon \mid x^2 \mid /2$ for all $n > N_0$, since $x^2 > 0$ for $x \neq 0$. Likewise, we can find N_1 such that $\mid x_n - x \mid < \mid x \mid /2$ for all $n > N_1$, so that $\mid x_n \mid > \frac{|x|}{2}$. Then, for $n > max\{N_0, N_1\}$:

$$|\frac{1}{x} - \frac{1}{x_n}| = |\frac{x - x_n}{xx_n}| < |\frac{\epsilon x^2}{2}\frac{2}{x^2}| = \epsilon$$

Therefore, since we chose ϵ arbitrarily, this is just the definition of a convergent sequence, so $(\frac{1}{x_1}, \frac{1}{x_2}, \ldots)$ converges to $\frac{1}{x}$.

4) Assume that the sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots)$ converges to $\mathbf{x}$. Then, for each $r > 0$, there exists an $N \in \mathbb{N}$ such that for $n > N$, $\|\mathbf{x} - \mathbf{x}_n\| < r$. However, by the Euclidean metric, this implies that:

$$\sqrt{(x_n^1 - x^1)^2 + \ldots + (x_n^k - x^k)^2} < r$$

for $n > N$. However, since all of these squares are non-negative, this implies that :

$$|x_n^j - x^j| < r$$

for all j and $n > N$. This is the definition of convergence of the sequences $(x_1^j, x_2^j, \ldots)$, so each of these sequences converges to x^j given that $\mathbf{x}_1, \mathbf{x}_2, \ldots)$ converges to $\mathbf{x}$.

For the converse, assume that each sequence $(x_1^j, x_2^j, \ldots)$ converges to x^j, and define $\mathbf{x} = (x^1, x^2, \ldots, x^k) \in \mathbb{R}^k$. Then, choose $\epsilon > 0$. Since each sequence $(x_1^j, x_2^j, \ldots)$ converges to x^j, we can find N_j such that $|x_n^j - x^j| < \epsilon/\sqrt{k}$ for all $n > N_j$. Then, if we let $N = max\{N_j\}$, for all $n > N$ we have:

$$\|x_n - x\| = \sqrt{(x_n^1 - x^1)^2 + \ldots + (x_n^k - x^k)^2} < \sqrt{k(\epsilon/k)} = \epsilon$$

Since we chose ϵ arbitrarily, this is the definition of convergence of $(\mathbf{x}_1, \mathbf{x}_2, \ldots)$ to $\mathbf{x}$.

5) (1) If a sequence if converging, then it is bounded.

True. Proof: Assume $(x_1, x_2, \ldots)$ converges to x. By (2), there are only a finite number of x_n outside of the open ball $S(x, 1)$. Let $R = max\|x_n\|$ for those x_n that fall outside of the $S(x, 1)$, which must exist since it is a finite set. Then, by the triangle inequality, $\|x_n\| < max(R, \|x\| + 1)$ for all x_n. Therefore, the set is bounded.

(2) If a sequence is bounded, then it is converging.

False. In $\mathbb{R}$, the sequence $(0, 1, 0, 1, 0, 1, \ldots)$ is clearly bounded as all elements are within $S(0, 2)$, but does not converge.

6) (1) A function $\mathbb{R}^{n,e} \to \mathbb{R}^{n,d}$ is continuous if and only if it is (locally) constant.

Proof: Assume some function $f : \mathbb{R}^{n,e} \to \mathbb{R}^{n,d}$ is continuous, and takes value y at some point x. Then, let $\epsilon = \frac{1}{2}$. By the definition of continuity, we can find $\delta > 0$ such that for all $z \in S(x, \delta)$ all points have $\|f(z) - y\| < \frac{1}{2}$. However, in the discrete metric, $\|a\| < \frac{1}{2}$ if and only if $a = 0$, since otherwise $\|a\| = 1$. So, in some neighborhood of x of radius δ, f must take the constant value y, so f is locally constant.

Note: Since $\mathbb{R}^n$ is connected, this indicates that f is constant everywhere. However, since we don't have a defintion of connectedness or a way to prove this, we will accept both the answers f is locally constant of f is constant.

(2) A function $\mathbb{R}^{n,d} \to \mathbb{R}^{n,e}$ is always continuous.

Proof: For any choice of $\epsilon > 0$, we can choose $\delta = \frac{1}{2}$. Since the only point in $S(x, \frac{1}{2})$ of $x \in \mathbb{R}^{n,d}$ is x itself, the definition of continuity will be satisfied by $\delta = \frac{1}{2}$ for any $\epsilon > 0$.