PROBLEM SET 4 SOLUTIONS

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1. Problem 1

1) For all $M > 0$, there exists a $\delta > 0$ such that whenever $|x - a| < \delta$, it is true that $|f(x)| > M$. In other words, for every $\frac{1}{M} > 0$, there is a $\delta > 0$ such that whenever $|x - a| < \delta$, it is true that $|f(x)| < \frac{1}{M}$. This completes the proof.

2) Similarly, for all $M > 0$, there exists a $\delta > 0$ such that whenever $|x - a| < \delta$, it is true that $|f(x)| < \frac{1}{M}$. In other, when x is within that range, it is true that $M < |f(x)|$. This completes the proof.

Note that we didn't really work with an ϵ. We wrote it as $\frac{1}{M}$ for out convenience. This is a very common thing to do, both to the ϵ and to the δ in the defition of continuity. In addition, note that δ is preserved in each proof. It is often the case that the δ we are looking for can be expressed as a combination of previous ones we had. You will find an example of this in Rudin, where he proves properties of continuous functions.

2. Problem 2

1) Note that

$$f(\phi(t)) = f(a, at) = \frac{2at}{a^2 + t^2} \leq \frac{2at}{a^2} = \frac{2t}{a}.$$

Pick $\epsilon > 0$ and define $\delta = \frac{a\epsilon}{2}$. Then whenever $0 < |t| < \delta$, it is true that

$$|f(\phi(t))| < \epsilon.$$

This proves that f is continuos at the origin on any straight line through it.

I should point out that a few brave souls actually showed the inequality the hard way and succeeded. Good job!

2) Note that if $t \neq 0$, then

$$f(\psi(t)) = f(t, t^2) = 1.$$

However, note that $f(\psi(0)) = 0$.

We're basically done,since f takes value 1 when it moves around the origin, where it has a little gap and takes on the value 0. We only need to turn this into math. To do this, we pick half the distance between 1 and 0 as our ϵ.

For $\epsilon = \frac{1}{2}$, we conclude that

$$|f(\psi(0)) - f(\psi(t))| > \frac{1}{2} = \epsilon.$$

This shows that f is not continuous at the origin.

This problem emphasizes the issue of direction when you study continuity in two or more variables. We require a function to be continuous in every possible direction when approaching a point. This makes continuity a bit more difficult to verify unless you can find good values for your function around the point that you are studying.

3. Problem 3

I won't draw the picture, but will tell you how to construct it.

1) We order the rational numbers in $[0,1]$ in this way:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \ldots.$$

Just draw a couple of points graphing the values of f in this way. If any of you is particularly interested, I could draw a picture (it's very cool).

2) Write a rational number $x = p/q$ in lowest terms. Then

$$f\left(\frac{p}{q} + \frac{p}{q^n}\right) = f\left(\frac{p(q^{n-1}+1)}{q^n}\right) = \frac{1}{q^n}.$$

Then

$$f\left(\frac{p}{q}\right) - f\left(\frac{p}{q} + \frac{p}{q^n}\right) = \frac{q-1}{q^2}$$

for any positive integer n. As n gets larger, this shows discontinuity at x.

3) Let a be an irrational number. Pick $\epsilon > 0$. We need to find a $\delta > 0$ such that whenever $|x - a| < \delta$, it is true that $|f(x) - f(a)| < \epsilon$.

Since a is irrational, this translates into finding $\delta > 0$ such that whenever $|x-a| < \delta$, it is true that $f(x) < \epsilon$. Since $f(y) = 0$ unless y is rational, we only need to worry about rational x.

Pick any $0 < \frac{p}{q} < \epsilon$ in lowest terms. Consider the set

$$S = \{\frac{r}{s} | \ 1 \geq \frac{r}{s} > 0, GCD(r,s) = 1, s \leq q\}.$$

Note that this set is finite, and label it $S = \{x_1, \ldots, x_n\}$. Now pick

$$\delta < min_i\{|a - x_i|\}.$$

It follows that whenever $|x - a| < \delta$, then $x \neq x_i$ for any i, so its denominator is less than q. Thus

$$f(x) < \frac{1}{q} < \epsilon,$$

and we conclude the proof.

Basically, this argument gets around the issue that I discussed with some of you in Office Hours and after my Section about how to know that the denominators get larger and larger. Although this construction seems a bit counterintuituve, it makes sense when you think about what it does: It rules out the finitely many fractions with small denominators that we need to rule out. Other good solutions are by contradiction or by directly evaluating some limits.

4. Problem 4

First note that the closed ball is closed because any closed ball in a metric space is closed (you showed this in pset 2). Also note that it is bounded because for any x in it, $d(x, 0) =< 2$. To show that it is not compact, we will show that there is an infinite sequence with no limit point.

Let $A = \{a_1, \ldots, a_k, \ldots\}$ be the infinite sequence (of sequences a_i) in which a_i is zero everywhere except at the i^{th} position, where it is 1. Since $d(a_i, 0) = 1$, the sequence is in $S[0, 1]$. Assume that A has a convergent subsequence $\{b_1, \ldots, b_k, \ldots\}$

to some point $K \in S[0,1]$. Then for any $\epsilon > 0$, there exists an $M > 0$ such that whenever $n > M$, it is true that

$$d(K, b_n) < \epsilon.$$

Then by the triangle inequality,

$$\sqrt{2} = d(b_n, b_m) \leq d(b_n, K) + d(K, b_m) = 2\epsilon.$$

But then $\epsilon > \sqrt{2}$ for any $\epsilon > 0$. This is, of course, false.
Hence, since limit points of a sequence give convergent subsequences, we conclude that A has no limit points. Now, since $S[0,1]$ has an infinite sequence with no limit points, it is not compact.
There three things I'd like to say about this problem.
First of all, note how it is more intimidating than difficult.
Second, note how we proved that it is not compact by using a fact about infinite subsequence in compact spaces. This was the natural thing to do since we are given a metric that seems to be related to sequences. Our choice of sequence is inspired by standard bases.
Finally, please note that this shows that Heine-Borel does not work on any metric space.

5. Problem 5

1) Since S has no limit points in K, then for any $q \in K$, there is an $\epsilon_q > 0$ such that $|x - q| > \epsilon_q$ for all $x \in S - q$. It follows that $S(q, e_q) \cap S$ is either q or $\{\}$ depending of whether $q \in S$ or $q \notin S$ respectively.

2) For each q, let $B_q = S(q, e_q)$. Note that the set of balls $\cup_q \{B_q\}$ is an open cover of S. Note that there is no finte subcover since the union of any finite set of balls $\{B_{q1} \ldots B_{qn}\}$ contains, by construction, a finite number of elements of the infinite set S.

3) We showed the contrapositive of the theorem statement, so our proof is complete.

This problem was a typical 'grind-through-definitions' kind of solution. The other way is significantly harder, and your next pset is all about that. However, this proof is necessary since we are linking our definition of compactness with the official definition of compactness. Please read Rudin's proof of why continuity preserves compactness to understand the power of being able to alternate between definitions.