1 Problem Set 5 – Solutions

1.1 Problem 1

We wish to show that $\bar{\mathbb{Q}} = \mathbb{R}$. We will show that $\mathbb{R} \subset \bar{\mathbb{Q}}$ and that $\bar{\mathbb{Q}} \subset \mathbb{R}$.

$\mathbb{R} \subset \bar{\mathbb{Q}}$: We need to show that every $x \in \mathbb{R}$ is either such that $x \in \mathbb{Q}$ or x is a limit point of $\mathbb{Q}$. So choose $x \in \mathbb{R} - \mathbb{Q}$, and choose $\epsilon > 0$. Now consider the open ball $S(x, \epsilon)$. Since in $\mathbb{R}$, open balls are open intervals, we have that $S(x, \epsilon) = (x - \epsilon, x + \epsilon)$. Now, recall from pset 1.2.5, that for any $a, b \in \mathbb{R}$ there exist infinitely many distinct rational numbers in (a, b). So we see that the order of the set $S(x, \epsilon) \cap \mathbb{Q}$ is infinite. But since $\epsilon > 0$ was arbitrary, we see that x is, by definition, a limit point of $\mathbb{Q}$, so $x \in \bar{\mathbb{Q}}$. Finally, since $x \in \mathbb{R}$ was arbitrary, we get that $\mathbb{R} \subset \bar{\mathbb{Q}}$.

$\bar{\mathbb{Q}} \subset \mathbb{R}$: Recall from pset 2.6.3 that $A \subset B \Rightarrow \bar{A} \subset \bar{B}$. Clearly, $\mathbb{Q} \subset \mathbb{R} \Rightarrow \bar{\mathbb{Q}} \subset \bar{\mathbb{R}}$. But recall from class that $\mathbb{R}$ is a closed set, and so (also from class) $\bar{\mathbb{R}} = \mathbb{R}$. So we get that $\bar{\mathbb{Q}} \subset \mathbb{R}$, as desired.

1.2 problem 2

We will show that $\mathbb{Q}^n$ is a countable dense subset of $\mathbb{R}^n$. Cearly, $\mathbb{Q}^n \subset \mathbb{R}^n$ since $\mathbb{Q}^n$ is defined specifically as those entries of $\mathbb{R}^n$ with rational coordinates. Also, from class $\#\mathbb{Q} = \#\mathbb{N} \Rightarrow \#\mathbb{Q}^n = \#\mathbb{N}^n$ (simply by inducting on n), and again by induction on n it is easily seen that $\#\mathbb{N}^n = \#\mathbb{N}$, and hence, $\#\mathbb{Q}^n = \#\mathbb{N}$, and so $\mathbb{Q}^n$ is countable. Now, we show that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$. To do this, we use the fact that if a is a contact point of A then there exists a sequence $\{a_k\} \subset A$ (meaning that each $a_k \in A$), such that $\lim a_k = a$. This fact is clear from the definition of a contact point, for if a is a contact point of A then either $a \in A$ or a is a limit point of A. So in the first case, we see that the sequence $\{a_k\}$ where each $a_k = a$ clearly converges to a and is such that $\{a_k\} \subset A$. In the second case, by definition of limit point there exists a sequence $\{a_k\} \subset A$ converging to a, and so our fact is proved. We now show that $\bar{\mathbb{Q}}^n = \mathbb{R}^n$. Again we show that each contains the other.

$\bar{\mathbb{Q}}^n \subset \mathbb{R}^n$: Just as before, $\mathbb{Q}^n \subset \mathbb{R}^n \Rightarrow \bar{\mathbb{Q}}^n \subset \bar{\mathbb{R}}^n = \mathbb{R}^n$.

$\mathbb{R}^n \subset \bar{\mathbb{Q}}^n$: Choose $\vec{x} = (x^1, \ldots, x^n) \in \mathbb{R}^n$. Now, recall from pset 3.4 that a sequence $\{\vec{x}_k\} \subset \mathbb{R}^n$ converges to $\vec{x} = (x^1, \ldots, x^n) \in \mathbb{R}^n$ if and only if $\{x_k^i\}$ converges to $x^i \in \mathbb{R}$ for all $i = 1, \ldots, n$. But from part 1, we know that $\mathbb{R} = \bar{\mathbb{Q}}$, and so (because of our fact) for all $x^i \in \mathbb{R}$ there exists a sequence $\{x_k^i\} \subset \mathbb{Q}$ with $\lim x_k^i = x^i$ for each $i = 1, \ldots, n$. Therefore, if we define $\{\vec{x}_k\}$ such that each $\vec{x}_k = (x_k^1, \ldots, x_k^n)$, clearly $\{\vec{x}_k\} \subset \mathbb{Q}^n$ (since each $x_k^i \in \mathbb{Q}$), and $\lim \vec{x}_k = \vec{x}$, and so we have constructed a sequence of rational vectors

converging to $\vec{x}$. Therefore, each $\vec{x} \in \mathbb{R}^n$ is a contact point of $\mathbb{Q}^n$ (since each is the limit point of some sequence), and so $\mathbb{R}^n \subset \bar{\mathbb{Q}}^n$, and we are done.

1.3   problem 3

Note I slightly modified the hint.

We need to show that every separable metric space has a countable base. By definition, a separable metric space is one which has countable dense subset. So assume that a metric space X is separable, and let $A \subset X$ be a countable dense subset. Then we have $\bar{A} = X$, and so every $x \in X$ is a contact point of A. This means that that for all $x \in X$ and $\epsilon > 0$, $S(x, \epsilon) \cap A \neq \emptyset$. Now define the set $B = \{S(a, \frac{1}{n}) | a \in A, n \in \mathbb{N}\}$. We will show that B is a countable base.

We show first that B is a base. Choose $x \in X$. We must show that for all open sets U_α which contain x, there exists some $S(a, \frac{1}{n}) \in B$ such that $x \in S(a, \frac{1}{n}) \subset U_\alpha$. So suppose $x \in U_\alpha$
$\Rightarrow x$ is an interior point of U_α (since U_α is open).
$\Rightarrow$ There exists some $\epsilon > 0$ such that $S(x, \epsilon) \subset U_\alpha$.
$\Rightarrow$ There exists some $n \in \mathbb{N}$ such that $\frac{2}{n} < \epsilon$ (pset 1.2.4).
$\Rightarrow S(x, \frac{2}{n}) \subset S(x, \epsilon) \subset U_\alpha$.
But we know that there exists some $a \in A$ so that $a \in S(x, \frac{1}{n})$ (since A is dense in X).
$\Rightarrow x \in S(a, \frac{1}{n})$, and $S(a, \frac{1}{n}) \in B$.
So if we can show that $S(a, \frac{1}{n}) \in U_\alpha$ then we will have shown that for all $x \in X$, and every open $U_\alpha \subset X$, such that $x \in U_\alpha$ there exists an open ball $S(a, \frac{1}{n}) \in B$ such that $x \in S(a, \frac{1}{n}) \subset U_\alpha$, and so we will have shown that B is a base. So it remains to be seen that $S(a, \frac{1}{n}) \subset U_\alpha$. However, this is a consequence of the triangle inequality.
We notice that $S(a, \frac{1}{n}) = \{y \in X | d(a, y) < \frac{1}{n}\}$.
$\Rightarrow y \in S(a, \frac{1}{n}) \Rightarrow d(y, x) \leq d(a, y) + d(a, x) < \frac{1}{n} + \frac{1}{n} = \frac{2}{n} < \epsilon$.
$\Rightarrow S(a, \frac{1}{n}) \subset S(x, \frac{2}{n}) \subset S(x, \epsilon) \subset U_\alpha$.
And so we have shown that B is a base.

We now show that B is countable. For every $a \in A$, define $B_a = \{S(a, \frac{1}{n}) | n \in \mathbb{N}\}$. First, notice that clearly, $\bigcup_{a \in A} B_a = B$. Also, notice that there exists a natural bijection $f : \mathbb{N} \to B_a$ which sends $f(n) = S(a, \frac{1}{n})$. This is clearly a bijection (you guys should be able to check injectivity and

surjectivity of this map by now). And so we see that $\#B_a = \#\mathbb{N}$, and so each B_a is countable. Finally, there are countably many B_a since we chose A to be a countable dense subset. Therefore, we see that B is a countable union of countable sets, and so from pset 2.2.1, B is countable. And we are done.

1.4 problem 4

1.4.1 Step 1:

Fix $\delta > 0$ and choose $x_1 \in K$, where K is compact. Now define the sequence $\{x_n\} \subset K$ so that $d(x_{n+1}, x_i) \geq \delta$ for every $i = 1, \ldots, n$. We will show that this process terminates after a finite number of steps. Define the set $A_\delta = \{x_n \in \{x_n\}\}$. Then if the process does not terminate after a finite number of steps then $A_\delta \subset K$ is an infinite subset. However, by construction for any $a, a' \in A_\delta$, $d(a, a') \geq \delta$, and so clearly for any $\epsilon < \delta$, we have that $S(a, \epsilon) \cap A_\delta - \{a\} = \emptyset$, and so clearly a_δ has no limit points, which contradicts the compactness of K. Therefore, a_δ must be a finite subset, and so the process for choosing x_{n+1} must terminate.

1.4.2 Step 2:

Now the claim is that K may be covered by finitely many open balls of radius δ. This is clear. Define the set $B_\delta = \{S(a, \delta) | a \in A_\delta\}$ (where A_δ is defined as in part 1, and is now known to be finite, say of order m). Then B_δ covers K. To see this, choose $x \in K$. Then if $x \notin B_\delta$, then $d(x, a) \geq \delta$ for all $a \in A_\delta$. But then by definition of A_δ, $d(x, x_n) \geq \delta$ for all $x \in \{x_n\}$, but then the process needn't have terminated. So $x = x_{m+1} \in \{x_n\} \Rightarrow x \in S_\delta \Rightarrow \#A_\delta \geq m + 1 \Rightarrow\Leftarrow$. Therefore, any $x \in K$ is also such that $x \in B_\delta$, hence B_δ gives a finite open cover of K and so only finitely many open balls of radius δ are needed to cover K.

1.4.3 Step 3:

Clearly, the sets in the above steps (A_δ and B_δ) are determined for each δ (so by changing δ we also change A_δ and B_δ). In this part we will consider the set A_δ for various δ. In particular, we will be interested in A_δ when $\delta = \frac{1}{n}$ for some $n \in \mathbb{N}$. Instead of writing $A_{1/n}$ we will write A_n for simplicity. Similarly for B_n. Now we define the set $A = \bigcup_{n=1}^{\infty} A_n$. We will show that A is a countable dense subset of K. First, note that it is clearly a subset, since each $a \in A_\delta$ is such that $a \in K$, for all $\delta > 0$. It is clearly countable, since

from step 1 and 2, we know that each A_δ is finite, hence countable, and so A is the countable union of countable sets, and so it is countable (from pset 2.2.1). Finally, we must show that that A is dense in K, or that for any $\epsilon > 0$ and any $x \in K$ there exists some $a \in A$ such that $d(x, a) < \epsilon$. So if we choose $\epsilon > 0$, then we know from pset 1.2.4 that we may find some $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. Also, we know that some finite collection of balls of radius $\frac{1}{n}$ may be used to cover K which means that x is in at least one such ball. But these balls are of radius $\frac{1}{n}$ and centered at $a \in A$ and so we have found our $a \in A$ such that $d(x, a) < \frac{1}{n} < \epsilon$ as desired. Therefore, A is a countable, dense, subset of K.

1.5   problem 5

Note This part was the hardest part of the pset.

Suppose $K \subset X$ is compact. Recall from problem 4, step 1, that for every $\delta > 0$ we may cover K with finitely many open balls of radius δ. In particular, we may do so for every $\delta = \frac{1}{n}$ when $n \in \mathbb{N}$. We define A_n, to be the set of the centers of those balls, when $\delta = \frac{1}{n}$. We define B_n to be the union of the balls when $\delta = \frac{1}{n}$. From part 4, each A_n is a finite set. So we have $A_n = \{x_1^n, \ldots, x_{k_n}^n\}$, such that $B_n = S(x_1^n, \frac{1}{n}) \cup \cdots \cup S(x_{k_n}^n, \frac{1}{n}) \supset K$. Now, define $A = \bigcup_{n \in \mathbb{N}} A_n$, $B = \bigcup_{n \in \mathbb{N}} B_n$. We will denote the elements of B using shorthand as $b_n^a = S(a, \frac{1}{n})$. Then, from problem 4, $A \subset K$ is a countable dense subset, and B is a countable base of K.

Now consider an arbitrary (not necessarily countable) open cover of K, say $C = \{U_\alpha | \alpha \in I\}$. We must show that C admits a countable subcover. We construct a subset of C in the following way. For every $b_n^a = S(a, \frac{1}{n}) \in B$, either there exists at least one $\alpha \in I$ such that $b_n^a \subset U_\alpha$, or there doesn't. If there does not exist such an $\alpha \in I$, then ignore this b_n^a. If there does exist at least one α, then arbitrarily select one. Define J to be the set of all selected α, and let $C^* = \{U_\gamma | \gamma \in J\}$. Clearly, $C^* \subset C$. Also, clearly C^* is countable, since each $b_n^a \subset B$ corresponds to at most one $U_\gamma \in C^*$, and B is known to be countable. All that remains is to show that C^* covers K.

Choose $x \in K$.
$\Rightarrow x \in U_\alpha$ for all some $\alpha \in I$.
$\Rightarrow$ There exists an $\epsilon > 0$ such that $S(x, \epsilon) \subset U_\alpha$.
But from pset 1.2.4, there exists $n \in \mathbb{N}$ such that $\frac{2}{n} < \epsilon$.
$\Rightarrow S(x, \frac{2}{n}) \subset S(x, \epsilon) \subset U_\alpha$.
Since $A \subset K$ is a dense subset, we know that for each $n \in \mathbb{N}$ there exists an

$a \in A$ such that $x \in b_n^a$. We now use the triangle inequality to show that $b_n^a \subset S(x, \frac{2}{n})$. Clearly, for all $y \in b_n^a$ we have that $d(x,y) \leq d(y,a) + d(x,a) < \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$

$\Rightarrow b_n^a \subset S(x, \frac{2}{n}) \subset U_\alpha$.

$\Rightarrow b_n^a \subset U_\alpha$ for at least one $\alpha \in I$.

$\Rightarrow$ There exists $\gamma \in J$ such that $b_n^a \in U_\gamma$.

But recall that $x \in b_n^a$.

$\Rightarrow x \in U_\gamma$, and since $x \in K$ was arbitrary. We have shown that C^* is a countable subcover of K, as desired, and we are done.

1.6   problem 6

Suppose that K is not compact*. Then there exists an open cover which has no finite subcover. Now, if K is compact, then we know from part 5 that every open cover of K has a countable subcover. So let $\{U_\alpha : \alpha \in I\}$ be an arbitrary open cover of K, and let $C = \{U_1, U_2 \ldots\}$ be a countable subcover of K. Now, define the sequence of sets $\{F_n\}$ where each $F_n = K - U_1 \cup \cdots \cup U_n$, by assumption each F_n is nonempty, since $F_n = \emptyset \Rightarrow K \subset U_1 \cup \cdots \cup U_n$ and so $U_1 \cup \cdots \cup U_n$ would be a finite subcover, which contradicts our original assumption. So each F_n is nonempty. Now, consider a point $x \in F_n$

$\Rightarrow x \in K$ and $x \notin U_1 \cup \cdots \cup U_n$

$\Rightarrow x \in (U_1 \cup \cdots \cup U_n)^c = U_1^c \cap \cdots \cap U_n^c$. So then clearly, $x \in U_1^c \cap \cdots \cap U_{n-1}^c$

$\Rightarrow x \in (U_1 \cup \cdots \cup U_{n-1})^c \Rightarrow x \in K - U_1 \cup \cdots \cup U_{n-1} \Rightarrow x \in F_{n-1}$. So we see that $K \supset F_1 \supset F_2 \supset \cdots$.

Now define a set $S = \{x_1, x_2, \ldots | x_n \in F_n\}$. Since each F_n is nonempty, $S \subset X$ is an infinite subset. We will show that if $x \in K$ then x may not be a limit point of S. Choose $x \in K$. Then since C is a countable cover of K, we know that $x \in \bigcup_{i=1}^\infty U_i \Rightarrow x \in U_n$ for some $n \in \mathbb{N}$. Therefore, since U_n is open, we know that there exists some $\epsilon > 0$ such that the $S(x, \epsilon) \subset U_n$. But then $U_n \cap F_n = \emptyset \Rightarrow S(x, \epsilon) \cap F_n = \emptyset$. So since each $x_n \in F_n$ and as shown above we have that $F_m \subset F_n$ for all $m \geq n$, we see that $x_m \in F_n$ for all $m \geq n$. But then $x_m \notin S(x, \epsilon)$ for all $m \geq n$. Therefore, at most $n-1$ elements of S are in $S(x, \epsilon)$, but clearly $n-1$ is finite, and so x is not a limit point of the set. Finally, since $x \in K$ was arbitrary, we see that the infinite set S does not have a limit point in K, that by definition K is not compact. Contradiction!

At long last, we have shown that K compact $\Rightarrow K$ compact* as desired.