Solutions to Problem Set 6

Problem 1

(a)

The curve in $\mathbb{R}^4$ defined by $\phi(t) = (a\cos\omega t, a\sin\omega t, b\omega t)$ is

$$\Phi = (t, a\cos\omega t, a\sin\omega t, b\omega t),\ \ t \in \mathbb{R}.$$

To differentiate a multi-component function we just differentiate each component separately, and so the tangent line L to Φ at $t = 2$ is

$$\begin{aligned} L(s) &= \Phi(2) + \Phi'(2) * s \\ &= (2, a\cos 2\omega, a\sin 2\omega, 2b\omega) + (1, -a\omega\sin 2\omega, a\omega\cos 2\omega, b\omega)s \end{aligned}$$

where $s \in \mathbb{R}$. In matrix notation this looks like

$$L(s) = \begin{bmatrix} 2+s \\ a\cos 2\omega - a\omega s\sin 2\omega \\ a\sin 2\omega + a\omega s\cos 2\omega \\ 2b\omega + b\omega s \end{bmatrix}$$

(b)

We want to show that there is no $\tau \in (0, \frac{2\pi}{\omega})$ such that $\phi(\frac{2\pi}{\omega}) - \phi(0) = \frac{2\pi}{\omega}\phi'(\tau)$.

$$\phi(2\pi/\omega) - \phi(0) = (a, 0, 2\pi b) - (a, 0, 0) = (0, 0, 2\pi b)$$

$$\frac{2\pi}{\omega}\phi'(\tau) = (-2\pi a\sin\omega\tau, 2\pi a\cos\omega\tau, 2\pi b)$$

In order for the left hand sides to be equal, $\sin\omega\tau = \cos\omega\tau = 0$, which is impossible, hence there is no τ which satisfies the above equation.

Problem 2

The plane tangent to the graph $\Phi(x,y) = (x, y, \sin(x-y), \cos(x+y))$ at $a = (\pi/4, \pi/4)$ is

$$
\begin{aligned}
T_a\Sigma &= \{\Phi(a) + \Phi'(a)v | v \in \mathbb{R}^2,\ t \in \mathbb{R}\} \\
&= \left\{ \begin{bmatrix} \pi/4 \\ \pi/4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial x}(\frac{\pi}{4}, \frac{\pi}{4}) & \frac{\partial f_1}{\partial y}(\frac{\pi}{4}, \frac{\pi}{4}) \\ \frac{\partial f_2}{\partial x}(\frac{\pi}{4}, \frac{\pi}{4}) & \frac{\partial f_12}{\partial y}(\frac{\pi}{4}, \frac{\pi}{4}) \\ \frac{\partial f_3}{\partial x}(\frac{\pi}{4}, \frac{\pi}{4}) & \frac{\partial f_3}{\partial y}(\frac{\pi}{4}, \frac{\pi}{4}) \\ \frac{\partial f_4}{\partial x}(\frac{\pi}{4}, \frac{\pi}{4}) & \frac{\partial f_4}{\partial y}(\frac{\pi}{4}, \frac{\pi}{4}) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \middle| (x,y) \in \mathbb{R}^2 \right\} \\
&= \left\{ \begin{bmatrix} \pi/4 \\ \pi/4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \middle| (x,y) \in \mathbb{R}^2 \right\} \\
&= \left\{ \begin{bmatrix} \pi/4 + x \\ \pi/4 + y \\ x - y \\ -x - y \end{bmatrix} \middle| (x,y) \in \mathbb{R}^2 \right\}
\end{aligned}
$$

Problem 3

(a)

Recall what each statement means:

C1: $F(x)$ is continuously differentiable in an open set $U \subset D$ containing a, namely the partial derivatives

$$\frac{\partial f}{\partial x_i}(a) = \lim_{t \to 0} \frac{F(a - te_i) - F(a)}{t}$$

exist for $i = 1, 2, \ldots, n$ and are continuous $\forall a \in U$.

C2: $F(x)$ is differentiable in an open set $U \in D$ containing a, namely $F'(a)$ exists $\forall a \in U$.

C3: $F(x)$ is diffentiable at $x = a$, namely there exists a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{F(a+h) - F(a) - A(h)}{\|h\|} = 0$$

. We call $A(x)$ the derivative of F at x.

C4: $F(x)$ is continuous at x =a means that $\lim_{x\to a} F(x) = F(a)$, and that F has all directional derivatives at a means that

$$\lim t \to 0 \frac{F(a+tv)-F(a)}{t}$$

exists $\forall v \in \mathbb{R}^n$. We denote this limit as $D_v F(a)$.

C5: $F(x)$ has all directional derivatives at $x = a$, see above.

C6: $F(x)$ has all partial derivatives at $x = a$, namely all partial derivatives exist, see above.

(b)

$$\begin{aligned} D_{e_1} f(x,y) &= \lim_{t\to 0} \frac{f((0,0)+t(1,0))-f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{\frac{t*0}{t-0}-0}{t} \\ &= \lim_{t\to 0} \frac{0}{t} \\ &= 0 \end{aligned}$$

and similarly

$$\begin{aligned} D_{e_2} f(x,y) &= \lim_{t\to 0} \frac{f(0,t)-f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{0}{t} \\ &= 0. \end{aligned}$$

(c)

First we show the existence of directional derivatives. Let $v = (v_1, v_2)$.

$$\begin{aligned} D_v f(0,0) &= \lim_{t\to 0} \frac{f((0,0)+tv) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{f(tv_1, tv_2) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{\frac{t^3 v_1^2 v_2}{t^4 v_1^4 + t^2 v_2^2} - 0}{t} \\ &= \lim_{t\to 0} \frac{v_1^2 v_2}{t^2 v_1^4 + v_2^2} \\ &= \begin{cases} 0 & v_2 = 0 \\ \frac{v_1^2}{v_2} & v_2 \neq 0 \end{cases} \end{aligned}$$

Since the limit exists for every $v \in \mathbb{R}^2$ all the directional derivatives exist. However the function is not continuous, as we have shown in a previous problem set.

(d)

We proceed similarly to the previous problem, and let $v = (v_1, v_2)$.

$$\begin{aligned} D_v f(0,0) &= \lim_{t\to 0} \frac{f((0,0)+tv) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{f(tv_1, tv_2) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{\frac{t^3 v_1^2 v_2}{t^2 v_1^2 + t^2 v_2^2} - 0}{t} \\ &= \lim_{t\to 0} \frac{v_1^2 v_2}{v_1^2 + v_2^2} \\ &= \frac{v_1^2 v_2}{v_1^2 + v_2^2} \\ &= f(v). \end{aligned}$$

To show that f is continuous at $(0,0)$ and its value is 0, we must show that $\lim_{x,y\to 0} f(x,y) = f(0,0) = 0$. The y in the numerator could be positive or negative, and instead of taking the cases separately we show that the limit of the absolute value of f goes to 0.

$$\begin{aligned} \lim_{x,y\to 0} |f(x,y)| &= \lim_{x,y\to 0} \left| \frac{x^2 y}{x^2+y^2} \right| \\ &\leq \lim_{x,y\to 0} \frac{x^2|y|}{x^2} \\ &= \lim_{x,y\to 0} |y| \\ &= 0 \end{aligned}$$

Thus f is continuous at $(0,0)$. From class we know that if f were differentiable, then the linear map described in part (a) would be the matrix of partial derivatives. So let's assume that f is differentiable at $(0,0)$; then its derivative would be the 1×2 matrix

$$\begin{bmatrix} D_{e_1}f(0,0) & D_{e_2}f(0,0) \end{bmatrix} = \begin{bmatrix} f(1,0) & f(0,1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

This implies that the directional derivative in all directions is zero, but we know from calculating the directional derivatives above that they are not all 0. Take for example $D_{(1,1)}f(0,0) = f(1,1) = 1/2$.

Note: Try to visualize this function to understand what is really going on and why it is not differentiable. The fact that $D_v f(0,0) = f(v)$ implies that the graph of f consists of straight lines going through the origin (since f is continuous there they must.) In the direction of the axes the lines have zero slope, but in the first two quadrants it they "go up" and in the other two quadrants they "go down". No matter how close to zoom into the origin they will never smooth out and so it does not look locally like a plane. This is similar to what happens at the cusp of a cone: it is continuous and all the partial derivatives exist, but it is not differentiable.

(e)

Following the hint, we first show that $\forall \frac{p}{q} \in \mathbb{Q}, |(\frac{p}{q})^2 - 2| \geq \frac{1}{q^2}$. Let's assume we could find integers p and q such that $|(\frac{p}{q})^2 - 2| < \frac{1}{q^2}$. Multiplying through by q^2, a positive number, we obtain $|p^2 - 2q^2| < 1$. Now the left hand side is a non-negative integer less than 1, which mens that it is 0. Rearranging the terms, $p^2 = 2q^2$ or $p = \pm\sqrt{2}q$, but no two rationals (let alone integers) can satisfy this beause $\sqrt{2}$ is an irrational number. We have arrived at a contradiction and thus $|(\frac{p}{q})^2 - 2| \geq \frac{1}{q^2}$ holds.

We can factor the left hand side of this relation, $|\frac{p}{q} - \sqrt{2}||\frac{p}{q} + \sqrt{2}|$, and divide both sides by the second factor to obtain

$$|\frac{p}{q} - \sqrt{2}| \geq \frac{1}{|\frac{p}{q} + \sqrt{2}|q^2}.$$

Now when $\frac{p}{q}$ is in the region where $|\frac{p}{q} + \sqrt{2}| < 3$ we are done, namely when $-3-\sqrt{2} < \frac{p}{q} < 3-\sqrt{2}$. This interval contains the ball $S = S(\sqrt{2}, 1/10)$ since $1/10 < 3 - 2\sqrt{2}$. We consider this region from now on. We can always do this, because differentiability at a point is a local property, so I can restrict the domain to any open set containing the point of interest. Another way to look at it, employing the $\epsilon - \delta$ definition: we can always take our $\delta < 1/10$, since if we find a $\delta > 1/10$ that works then any $\delta < 1/10$ will certainly work too.)

Within our ball S we have

$$\frac{|f(\frac{p}{q}) - f(\sqrt{2})|}{|\frac{p}{q} - \sqrt{2}|} \leq \frac{|\frac{1}{q^3} - 0|}{\frac{1}{3q^2}} = \frac{3}{q}.$$

To show that f is differentiable at $\sqrt{2}$, we must show that $\forall \epsilon > 0,\ \exists \delta > 0$ such that

$$0 < h < \delta \Rightarrow \frac{|f(\sqrt{2}+h) - f(\sqrt{2})|}{|h|} = \left|\frac{f(\sqrt{2}+h)}{h}\right| \leq \epsilon.$$

If $\sqrt{2} + h$ is irrational then $f(\sqrt{2}+h) = 0$ and the so the relation works for every ϵ. Consider the case where $\sqrt{2} + h = p/q \in \mathbb{Q}$. Then

$$\left|\frac{f(\sqrt{2}+h)}{h}\right| = \frac{|f(\frac{p}{q}) - f(\sqrt{2})|}{|\frac{p}{q} - \sqrt{2}|} \leq \frac{3}{q}.$$

Using a similar argument as on a previous problem set, for a given ϵ there are only finitely many q such that $\frac{1}{q} > \frac{\epsilon}{3}$, and thus only finitely many $\frac{p}{q} \in S$ such that $\frac{1}{q} > \frac{\epsilon}{3}$. Thus we can choose a δ such that none of these are in the ball $S' = S(\sqrt{2}, \delta)$. Within S' then we have

$$\left|\frac{f(\sqrt{2}+h|}{h}\right| \leq \frac{3}{q} < \epsilon$$

and we are done.

Now we show that f is not continuous on any open interval containing $\sqrt{2}$ using the sequence definition of continuity. Consider a sequence $\{x_i\}$ of irrational numbers converging to $\frac{p}{q} \in \mathbb{Q}$. The image of the sequence $\{f(x_i)\}$ converges to 0 since $f(x_i) = 0, \ \forall i \in \mathbb{N}$. But $f(\frac{p}{q}) = \frac{1}{q^3} \neq 0$, so f is discontinuous at every rational point. Since the rationals are dense in $\mathbb{R}$, f is not continuous on any open interval, in particular no one that contains $\sqrt{2}$.

(f)

If f is differentiable then its derivative is the matrix of partial derivatives. Thus we will find this matrix and show that it does satisfy the conditions necessary.

$$
\begin{aligned}
\frac{\partial f}{\partial x}(0,0) &= \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x} \\
&= \lim_{x \to 0} \frac{x^2 \sin \frac{1}{x} - 0}{x} \\
&= \lim_{x \to 0} x \sin \frac{1}{x} \\
&= 0
\end{aligned}
$$

since $\sin \frac{1}{x}$ oscillates between -1 and 1 and x goes to 0. Similarly for the other partial derivative,

$$
\begin{aligned}
\frac{\partial f}{\partial y}(0,0) &= \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y} \\
&= \lim_{y \to 0} \frac{y^2 - 0}{y} \\
&= \lim_{y \to 0} y \\
&= 0.
\end{aligned}
$$

Thus the linear map, if it exists, is $A = \begin{bmatrix} 0 & 0 \end{bmatrix}$. Now we check to see if it works in the definition. We have to look at the cases where $h_1 \neq 0$ and

$h_1 = 0$ separately. First $h_1 \neq 0$:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(0+h) - f(0) - Ah}{\|h\|} &= \lim_{h\to 0} \frac{|h_1^2 \sin\frac{1}{h_1} + h_2^2|}{\sqrt{h_1^2 + h_2^2}} \\
&\leq \lim_{h\to 0} \frac{h_1^2 + h_2^2}{\sqrt{h_1^2 + h_2^2}} \\
&= \lim_{h\to 0} \|h\| \\
&= 0.
\end{aligned}$$

When $h = 0$ the limit reduces to a one-dimensional limit and we have

$$\begin{aligned}
\lim_{h\to 0} \frac{f(0+h) - f(0) - Ah}{\|h\|} &= \lim_{h_2\to 0} \frac{h_2^2}{h_2} \\
&= \lim_{h_2\to 0} h_2 \\
&= 0.
\end{aligned}$$

Thus the limit is well defined and f is differentiable at $(0,0)$, with $f'(0,0) = A = \begin{bmatrix} 0 & 0 \end{bmatrix}$.

Finally, we must show that the partial derivates are not continuous at $(0,0$. We look at $\partial f/\partial x$ and want to show that $lim_{x,y\to 0} \partial f \partial x(x,y) \neq \frac{\partial f}{\partial x}(0,0) = 0$. One counterexample suffices, and so we can approach $(0,0)$ along the $x-$axis. Thus the limit become

$$\begin{aligned}
\lim_{x\to 0} \frac{\partial f}{\partial x}(x,0) &= \lim_{x\to 0}(2x\sin\frac{1}{x} + x^2\frac{-1}{x^2}\cos\frac{1}{x}) \\
&= \lim_{x\to 0}(2x\sin\frac{1}{x} - \cos\frac{1}{x}) \\
&= -\lim_{x\to 0}\cos\frac{1}{x}
\end{aligned}$$

This limit does not exist because $\cos\frac{1}{x}$ has infinitely many oscillations between 1 and -1 in any neighborhood of 0. And we are done.