Problem Set 7 Solutions

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1 Problem 1

1.1

(a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable and homogeneous. Since it is homogeneous, $f(0) = 0f(x) = 0$. Since f is differentiable, all directional derivatives derivatives exist at 0, and $D_v f(0) = f'(0)v$ for all $v \in \mathbb{R}^n$. Then:

$$D_v f(0) = \lim_{t \to 0} \frac{f(0 + tv) - f(0)}{t} = \lim_{t \to 0} \frac{f(tv)}{t} = \lim_{t \to 0} \frac{tf(v)}{t} = f(v)$$

However, since $D_v f(0) = f'(0)v$, this implies that $f(v) = f'(0)v$.

1.2

(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous function.

$$D_v f(0) = \lim_{t \to 0} \frac{f(0 + tv) - f(0)}{t} = \lim_{t \to 0} \frac{f(tv)}{t} = \lim_{t \to 0} \frac{tf(v)}{t} = f(v)$$

as in part (a), so all directional derivatives must exist.

However, if $f'(0)$ exists, then it must be a linear transformation. However, as shown in (a), if f is differentiable at 0 then $f'(0)v = D_v f(0) = f(v)$, which implies that f must be a linear transformation as well.

1.3

(c) Let $f(x, y)$ be defined as in the problem. For any $v = (v_x, v_y) \neq (0, 0)$,

$$D_v f(0,0) = \lim_{t\to 0} \frac{f(tv) - f(0,0)}{t} = \lim_{t\to 0} \frac{t^3 v_x v_y^2}{t^3(v_x^2 + v_y^2)} = \frac{v_x v_y^2}{v_x^2 + v_y^2}$$

If $v_x, v_y = (0, 0)$, then $D_v f(0, 0) = 0$ by definition. So, all directional derivatives of f at $(0, 0)$ exist. However, f is not differentiable at 0 unless the map $v \to f'(0)v$ is a linear transformation. This implies that

$$D_{(v_x,v_y)} f(0,0) = v_x D_{(1,0)} f(0,0) + v_y D_{(0,1)} f(0,0)$$

But $D_{(1,0)} f(0, 0) = D_{(0,1)} f(0, 0) = 0$, whereas $D_{(1,1)} f(0, 0) = \frac{1}{2} \neq 0$, so f is not differentiable.

2 Problem 2

2.1

Let $x = r\cos\theta$ and $y = r\sin\theta$, and $g(r, \theta) = f(r\cos\theta, r\sin\theta$.

$$\frac{\partial g}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

$$\frac{\partial g}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial f}{\partial x} r\sin\theta + \frac{\partial f}{\partial y} r\cos\theta$$

$$(\frac{\partial g}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial g}{\partial \theta})^2 =$$

$$(\frac{\partial f}{\partial x})^2(\cos^2\theta + \sin^2\theta) + (\frac{\partial f}{\partial y})^2(\cos^2\theta + \sin^2\theta) + (\frac{\partial f}{\partial x})(\frac{\partial f}{\partial y})(2\cos\theta\sin\theta - 2\cos\theta\sin\theta) =$$

$$(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2$$

2.2

(b) Let $g(r, \theta, z) = f(r\cos\theta, r\sin\theta, z)$, still with $x = r\cos\theta$ and $y = r\sin\theta$. By (a) we have

$$(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 = (\frac{\partial g}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial g}{\partial \theta})^2$$

So, since z is independent of x, y, r, and θ, and f and g have identical dependence on z:

$$(\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 + (\frac{\partial f}{\partial z})^2 = (\frac{\partial g}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial g}{\partial \theta})^2 + (\frac{\partial g}{\partial z})^2$$

2.3

(c) Let $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\cos\theta, z = \rho\cos\phi$, and let $h(\rho, \theta, \phi) = f(\rho\sin\phi\cos\theta, \rho\sin\phi\cos\theta, \rho\cos\phi)$.

$$\frac{\partial h}{\partial \theta} = \frac{\partial f}{\partial x}\rho\cos\phi\cos\theta + \frac{\partial f}{\partial y}\rho\cos\phi\sin\theta - \frac{\partial f}{\partial z}\rho\sin\phi$$

SORRY STILL UNDER CONSTRUCTION; THIS WILL BE FINISHED REALLY SOON! Basically, you should just make sure to carefully use the chain rule to take all partial derivatives, and make sure to use the product rule ($\frac{\partial}{\partial x}(ab) = b\frac{\partial a}{\partial x} + a\frac{\partial b}{\partial x}$ wherever applicable).

3 Problem 3

3.1

(a) Taylor series of $\sin(x)$ around $x = \frac{\pi}{4}$:

$$\sin\frac{\pi}{4} + \frac{\sqrt{2}}{2}(x - \frac{\pi}{4}) - \frac{1}{2}\frac{\sqrt{2}}{2}(x - \frac{\pi}{4})^2 - \frac{1}{6}\frac{\sqrt{2}}{2}(x - \frac{\pi}{4})^3 + \frac{1}{24}\frac{\sqrt{2}}{2}(x - \frac{\pi}{4})^4 + \ldots$$

To compute the first four digits of $\sin 50°$ we simply keep adding term from above until the first four digits are fixed. We have the correct digits, 0.7660 after substituting in these 5 terms.

3.2

(b) $\frac{d^n}{dx^n} log(x) = (-1)^{n+1}(n-1)!\frac{1}{x^n}$, as can be seen from repeated differentiation. So, the Taylor series of $log(x)$ around $x = 1$ is:

$$\sum_n (-1)^{n+1}(n-1)!\frac{(x-1)^n}{n!} = \sum_n (-1)^{n+1}\frac{(x-1)^n}{n}$$

So, at $x = \frac{3}{2}$, we have

$$\log(\frac{3}{2}) = \sum_n (-1)^{n+1}\frac{1}{n2^n} \approx 0.406$$

where we can calculate the above number by substituting in successive terms from the Taylor expansion.

3.3

(c) Claim:

$$\frac{d^n}{dx^n}(1+x)^\alpha = \alpha(\alpha-1)(\alpha-2)\cdot\ldots\cdot(\alpha-n+1)(1+x)^{\alpha-n}$$

We will show this by induction. For $n = 0$ it reduces to $(1+x)^\alpha = (1+x)^\alpha$, so the claim holds for $n = 0$. Assuming it holds for $n = k$:

$$\frac{d^k}{dx^k}(1+x)^\alpha = \alpha(\alpha-1)(\alpha-2)\cdot\ldots\cdot(\alpha-k+1)(1+x)^{\alpha-k}$$

$$\frac{d^{k+1}}{dx^{k+1}}(1+x)^\alpha = \alpha(\alpha-1)(\alpha-2)\cdot\ldots\cdot(\alpha-k+1)(\alpha-k)(1+x)^{\alpha-k-1}$$

through standard differentiation. So, the claim holds for all $n \in \mathbb{N}$ by induction.

Using this, we can calculate the Taylor series around $x = 0$:

$$\sum_{n=0}^{\infty} \alpha(\alpha-1)\cdot\ldots\cdot(\alpha-n+1)\frac{x^n}{n!} = \binom{\alpha}{n}x^n$$

Note that $\left|\binom{\alpha}{n}\right| = \left|\frac{\alpha}{1}\frac{\alpha-1}{2}\cdot\ldots\cdot\frac{\alpha-n+1}{n}\right|$. Since $|\alpha-n+1| < n$ for $n > \alpha+1$ we can conclude that $\binom{\alpha}{n}$ is bounded over all $n \in \mathbb{N}$, so $\binom{\alpha}{n} < M$ for some $M \in \mathbb{R}$. Then:

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n < M \sum_{n=0}^{\infty} x^n$$

which clearly converges for $|x| < 1$ since it is a geometric series. So, the Taylor expansion converges for all $x \in \mathbb{R}$ for $|x| < 1$.

4 Problem 4

Assume f and g are continuous k-times differentiable functions and that f, g and their first $k-1$ first derivatives vanish at $a \in \mathbb{R}$. Then, using the Taylor theorem, in some neighborhood of a, we have $f(a+h) = f^{(k)}(a)\frac{h^k}{k!} + R_f(h)$ where $\frac{R_f^h}{h^k} \to 0$ as $h \to 0$, and similarly for $g(a+h)$. So:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\frac{f^{(k)}(a)(x-a)^k}{k!} + R_f(x-a)}{\frac{g^{(k)}(a)(x-a)^k}{k!} + R_g(x-a)}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f^{(k)}(a) + k!\frac{R_f(x-a)}{(x-a)^k}}{g^{(k)}(a) + k!\frac{R_g(x-a)}{(x-a)^k}}$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f^{(k)}(a)}{g^{(k)}(a)}$$

since both remainder terms go to 0 as $x \to a$.