

## SOLUTIONS FOR PROBLEM SET 8

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### 1. PROBLEM 1

a) You may calculate the first and second partial derivatives to see that they are of the form  $cx^{a_1}y^{a_2}z^{a_3}$ , where  $c$  is a constant and the  $a_i$  are integers less than three. After evaluating them at  $(1,0,-1)$ , you will find that the only non-vanishing one is  $\frac{\partial^2 f}{\partial y^2}$ , in which case we obtain  $-2$ . Hence the Taylor polynomial up to second order for  $h = (h_1, h_2)$  is simply  $f(h) = -h_2^2$ .

b) We may compute  $R_2(h) = f(a+h) - P_2(h)$ . Plugging in the values of  $P_2$  and  $f(a+h)$ , we find that

$$\frac{R_2(h)}{\|h\|^2} = \frac{h_2^2((1+h_1)(-1+h_3)^3 + 1)}{\|h\|^2}.$$

In addition, note that

$$-\frac{h_2^2|h_2^2((1+h_1)(-1+h_3)^3 + 1)|}{h_2^2} \leq \frac{h_2^2(h_2^2((1+h_1)(-1+h_3)^3 + 1))}{h_1^2 + h_2^2 + h_3^2}.$$

Similarly,

$$\frac{h_2^2(h_2^2((1+h_1)(-1+h_3)^3 + 1))}{h_1^2 + h_2^2 + h_3^2} \leq \frac{h_2^2|h_2^2((1+h_1)(-1+h_3)^3 + 1)|}{h_2^2}.$$

It follows that

$$\left| \frac{h_2^2(h_2^2((1+h_1)(-1+h_3)^3 + 1))}{h_1^2 + h_2^2 + h_3^2} \right| \leq |h_2^2((1+h_1)(-1+h_3)^3 + 1)|,$$

from which we see that

$$\left| \frac{R_2(h)}{\|h\|^2} \right| \leq |h_2^2((1+h_1)(-1+h_3)^3 + 1)|.$$

Now we can take the limit as  $h \rightarrow 0$ , and complete the computation.

### 2. PROBLEM 2

a) Equating the Jacobian to zero, we find that the only critical point of  $f$  is  $(1, 1)$ . The Hessian at this point is the matrix  $H$  whose rows are  $(4, -1)$  and  $(-1, -6)$ . The upper left  $1 \times 1$  matrix has determinant 4. The matrix itself has determinant  $-25$ . Hence the matrix is not positive definite. Similarly, doing this for  $-H$  shows that the matrix is not negative definite. Since its determinant is nonzero,  $(1, 1)$  is a saddle point.

b) Setting the Jacobian equal to zero and solving the corresponding system of equations, we find that the critical points are  $(0,0)$ ,  $(2,3)$  and any points of the form  $(x, 0)$  or  $(0, y)$ . The Hessian of this function is the two by two matrix of whose rows are  $(12y^2 - 6xy^3 - 2y^4, 36xy^2 - 9x^2y^2 - 8xy^3)$  and  $(36xy^2 - 9x^2y^2 - 8xy^3, 36x^2y - 6x^3y - 12x^2y^2)$ .

Hence,  $H_{2,3}$  has rows  $(-162, -108), (-108, -144)$ . This is negative definite, so the point is a maximum. When  $x = 0$  or  $y = 0$ , the determinants are zero, so the test is not conclusive.

c) The Jacobian has row  $(\cos(x)\cosh(y), \sin(x)\sinh(y))$ . The Hessian has rows  $(-\sin(x)\cosh(y), -\cos(x)\sinh(y)), (-\cos(x)\sinh(y), \sin(x)\cosh(y))$ . Setting the Jacobian equal to zero, we find that the critical points are those of the form  $(\frac{\pi}{2} + 2\pi k, 2\pi k)$ . At these points, the Hessian has rows  $(-1, 0), (0, 1)$ . We see that this is a local maximum.

d) The Jacobian has row  $(2e^{x^2-y^2}x(1+x^2+y^2), 2e^{x^2-y^2}y(1+x^2-y^2))$ . Setting this equal to zero, we find the critical points:  $(0, 0), (0, -1)$  and  $(0, 1)$ .

- The Hessian for  $(0, 0)$  has rows  $(2, 0), (0, 2)$ . From it,  $(0, 0)$  is a minimum.
- The Hessian for  $(0, -1)$  and  $(0, 1)$  has rows  $(4e^{-1}, 0), (0, -4e^{-1})$ . From it,  $(0, -1)$  and  $(0, 1)$  are saddle points.

### 3. PROBLEM 3

(Sketch)The area of the triangle is  $\frac{\sqrt{3}x^2}{36}$ . The area of the rectangle is  $\frac{y^2}{9}$ , and the area of the square is  $\frac{(100-x-y)^2}{16}$ . Hence the total area is given by

$$A(x, y) = \frac{90000 + (9 + 4\sqrt{3})x^2 + 17y^2 + 18xy - 1800(x + y)}{144}.$$

Setting its Jacobian equal to zero, we find the critical point  $(\frac{1800}{18+17\sqrt{3}}, \frac{900\sqrt{3}}{18+17\sqrt{3}})$ . At this point, the Hessian has rows  $(9 + 4\sqrt{3}, 9), (9, 17)$ . We conclude that the point is a local minimum. However, our computation is not done. We must consider the special cases  $x = 0, y = 0$  and  $100 - x - y = 0$ . Plugging these equations into the definition of  $A(x, y)$  and setting the Jacobian equal to zero, we find the critical points  $(0, 900), (\frac{900}{9+4\sqrt{3}}, 0)$  and  $(\frac{250}{2+\sqrt{3}}, \frac{100\sqrt{3}-50}{2+\sqrt{3}})$ . In addition, we must compute the values at the degenerate cases  $(0, 0), (100, 0)$  and  $(0, 100)$ . Evaluating  $A$  at all the possibilities, we find that  $(\frac{1800}{18+17\sqrt{3}}, \frac{900\sqrt{3}}{18+17\sqrt{3}})$  is the global maximum.

### 4. PROBLEM 4

a) The Jacobian has row  $(12x^3 - 8xy, -4x^2 + 2y)$ . Setting it equal to zero and solving, we find that  $(0, 0)$  is a critical point.

b) Substitute  $y = mx$ . Then  $f(x, y)$  becomes a one variable function  $g(x) = f(x, mx) = 3x^4 - 4mx^3 + m^2x^2$ . Then  $g'(x) = 12x^3 - 12mx^2m^2x$  and  $g''(x) = 36x^2 - 24mx + 2m^2$ . Plugging in  $x = 0$ , we find that  $g''(0, 0) > 0$ , so that the function has a local minimum at the origin.

When  $m = 0$ , this is simply a quadratic on  $x$  with a concave up graph with vertex at the origin, so the result follows.

c) For any small distance  $\epsilon > 0$ , there is a point of the form  $(b, b^2)$  at which  $f$  is negative. Here,  $f(b, b^2) = -b^4$ . Note that  $g(b) = (b, b^2)$  is a continuous map that tends to 0 as  $b \rightarrow 0$ .

## 5. PROBLEM 5

We may rewrite  $E(a, b)$  as

$$E(a, b) = a^2 \sum_{i=1}^n x_i^2 + 2ab \sum x_i - 2a \sum x_i y_i + nb^2 + ny^2 - 2b \sum y_i.$$

By setting its Jacobian equal to zero, we find the critical point

$$(a, b) = \left( \frac{n \sum x_i y_i - (\sum x_i)(\sum y_i)}{n \sum x_i^2 - (\sum x_i)^2}, \frac{(\sum x_i^2) \sum y_i - (\sum x_i) \sum x_i y_i}{n (\sum x_i^2 - (\sum x_i)^2)} \right).$$

The Hessian has rows  $(2 \sum x_i^2, 2 \sum x_i), (2 \sum x_i, 2n)$ . The upper left one by one matrix has determinant  $2 \sum x_i^2$  which, under the hypothesis given, is positive. The matrix has determinant  $4n \sum x_i^2 - 4 (\sum x_i)^2$ . Note that this is always positive because of the Cauchy-Schwarz inequality from last semester. A special case of this inequality states that the product of the square of the norms of two vectors is greater or equal to the square of the dot product between them, with equality if and only if the vectors are linearly dependent. In coordinates, this reads

$$\left( \sum x_i^2 \right) \left( \sum y_i^2 \right) \geq \left( \sum x_i y_i \right)^2.$$

Setting all the  $y_i = 1$ , we obtain the inequality we are looking. We conclude by noting that we will not get an equality, since all the  $x_i$  are different from each other, which prevents linear dependence with the vector  $(1, 1, 1, 1, \dots, 1)$ .