Solutions to Problem Set 10

Problem 1

Let $f,\ g : \mathbb{R}^n \to \mathbb{R}$ be admissible functions and let f be continuous on U and g continuous on V. From the definition of admissible, we know that $vol(\mathbb{R}^n \setminus U) = vol(U^c) = 0$ and $vol(V^c) = 0$.

(i)$f + g$ is admissible:

The set on which both f and g, and thus $f + g$, are continuous is $\mathbb{R}^n \setminus (U^c \bigcup V^c)$. So, to show that $f + g$ is admissible we must show that $vol(U^c \bigcup V^c) = 0$.

We know from class that $0 \leq vol(U^c \bigcup V^c) \leq vol(U^c) + vol(V^c) = 0$.

(ii)cf is admissible, where $c \in \mathbb{R}$:

cf is continuous at the points where f is continuous, and so the set of points on which cf is not continuous is also U^c, and we already know that $vol(U^c) = 0$.

Problem 2

For a function f, we define

$$f^+(x) = \begin{cases} f(x) & f(x) \geq 0 \\ 0 & f(x) < 0 \end{cases},\ f^-(x) = \begin{cases} f(x) & f(x) \leq 0 \\ 0 & f(x) > 0 \end{cases}$$

so that $f(x) = f^+(x) - f^-(x)$. We say that f is integrable iff f^+ and f^- are integrable and

$$\int_A f dx = \int_A f^+ dx - \int_A f^- dx.$$

Then $|\int_A f(x)| = |\int_A f^+ dx - \int_A f^- dx|$. Now we define

$$|f(x)| = \begin{cases} f(x) & f(x) \geq 0 \\ -f(x) & f(x) < 0 \end{cases}$$

namely $|f(x)| = f^+(x) + f^-(x)$. Thus we have

$$\begin{aligned} \int_A |f(x)| &= \int_A f^+(x) + \int_A f^-(x) \\ &\leq \left| \int_A f^+(x) - \int_A f^-(x) \right| \\ &= \left| \int_A f(x) \right| \end{aligned}$$

Problem 3

To show that f is integrable, we must show that O_f is measurable in $\mathbb{R}^N$, namely we must find two sequences of polygons P_i and Q_i such that $P_i \subset O_f \subset Q_i$ and $\forall \epsilon > 0$, $\exists N$ such that $vol(Q_i) - vol(P_i) < \epsilon$ for $i > N$. The Q_i's are not so important since we're going to show that $vol(O_f) = 0$, so let's focus on constructing the P_i's from rectangles as follows. Given i, choose the first rectangle to be $[0,1] \times [0,1] \times [0, 1/i]$. The points whose image is not in this rectangle are infinitely may, but they are arranged in such a way that we can cover them with a finite number of volume-zero rectangles. There are only a finite number of y-values of points in the domain whose image is not in the first rectangle, namely $y = p/q$ where $p \leq q < i$. If $y_1, \cdots, y_k$ are these values, then the retcangles $[0,1] \times [y_1, y_1] \times [0,1], \cdots, [0,1] \times [y_k, y_k] \times [0,1]$ contain the remainder of O_f, there are of course finitely many of them, and they have volume zero.

Thus if P_i is the union of these rectangles, then $vol(P_i) = 1 * 1 * 1/i = 1/i$. This gets arbitrarily small for large enough i, and so $vol(O_f) = 0$.

Problem 4

It suffices to show one direction, since the other way is exactly the same. So, start with f integrable on A. If A is measurable and C is measurable, then $A \backslash C$ is measurable. We first show that any bounded function which is non-zero only on a set of measure zero (C) is integrable with volume zero. This is because $O_{any\ such\ function} \subset C \times [-M, M]$, and

$$vol_{n+1}(C \times [-M, M]) = vol_n(C) * (2M) = 0.$$

An obvious corollary is that any two such functions have the same volume (both zero).

Now back to our proof. We have

$$\begin{aligned}
\int_A f &= \int_{A\setminus C} f + \int_C f \\
&= \int_{A\setminus C} g + \int \chi_C f \\
&= \int_{A\setminus C} g + 0 \\
&= \int_{A\setminus C} g + \int \chi_C g \\
&= \int_{A\setminus C} g + \int_C g \\
&= \int_A g
\end{aligned}$$

which is what was to be shown.

Problem 5

Here I assume that $f(x)$ is positive. The idea is the same for the case where $f(x)$ is negative or zero, you just have to be careful when defining P_1 and P_2.

Define $f_{min} = \inf\{f(x)|x \in Q_\epsilon\}$ and $f_{max} = \sup\{f(x)|x \in Q_\epsilon\}$, and consider the polygons $P_1 = Q_\epsilon \times [0, f_{min}]$ and $P_2 = Q_\epsilon \times [0, f_{max}]$. (I can always choose an ϵ small enough so that f_{min} is positive because f is continuous.) Then since $P_1 \subseteq O_f \subseteq P_2$, we have

$$\begin{aligned}
vol(P_1) \leq \quad & vol(O_f) \quad \leq vol(P_2) \\
vol(Q_\epsilon) f_{min} \leq \quad & \int_{Q_\epsilon} f \quad \leq vol(Q_\epsilon) f_{max} \\
f_{min} \leq \frac{1}{vol(Q_\epsilon)} & \int_{Q_\epsilon} f \leq f_{max}.
\end{aligned}$$

Note that the volumes in the first line are $n+1$-dimensional whereas $vol(Q_\epsilon)$ is an n-dimensional volume. Since f is continuous, f_{min} and f_{max}

get arbitrarily close to each other and thus also to $f(x)$, for small enough ϵ. Thus we have shown that

$$\lim_{\epsilon \to 0} \frac{1}{vol(Q_\epsilon)} \int_{Q_\epsilon} f = f(x).$$

Problem 6

We have a bounded increasing function $f : [a, b] \to \mathbb{R}$ and we want to show that f is integrable. We start by partitioning the domain into n equal intervals, I_k, each of length $vol(I_k) = \frac{b-a}{n}$, and let the endpoints of these intervals be $a = c_0, c_1, \cdots, c_n = b$. We define two polygons $P_n,\ Q_n$ as follows:

$$P_n = \bigcup_k I_k \times \inf\{f(x)|x \in I_k\} = \bigcup_k I_k \times f(c_{k-1})$$

$$Q_n = \bigcup_k I_k \times \sup\{f(x)|x \in I_k\} = \bigcup_k I_k \times f(c_k).$$

It is clear that $P_n \subset O_f \subset Q_n$, so now we have to show that $vol(Q_n) - vol(P_n)$ gets arbitrarily small for large enough n. We use the fact that the rectangles in the unions above are disjoint, so their volumes just add.

$$\begin{aligned} vol(Q_n) &= vol(\bigcup_k I_k \times f(c_k) \\ &= \sum_k vol(I_k \times f(c_k)) \\ &= \sum_{k=1}^{n} \frac{b-a}{n} * f(c_k) \\ &= \frac{b-a}{n} \sum_{k=1}^{n} f(c_k) \end{aligned}$$

and similarly

$$\begin{aligned} vol(P_n) &= \frac{b-a}{n} \sum_{k=1}^{n} f(c_{k-1}) \\ &= \frac{b-a}{n} \sum_{k=0}^{n-1} f(c_k). \end{aligned}$$

Now we subtract these and see that most of the terms cancel:

$$\begin{aligned}
vol(Q_n) - vol(P_n) &= \frac{b-a}{n}\sum_{k=1}^{n} f(c_k) - \frac{b-a}{n}\sum_{k=0}^{n-1} f(c_k) \\
&= \frac{b-a}{n}(f(c_n) - f(c_0) \\
&= \frac{b-a}{n}(f(b) - f(a)).
\end{aligned}$$

As $n \to \infty$ this quantity goes to 0 because of the n in the denominator, and we are done. $\ddot\smile$