

Problem 1: (a) Since f is integrable, $|f|$ is integrable, as we showed on the last problem set. Therefore, we can assume without loss of generality that f is always positive. Since f is integrable, as proven in class, given $\epsilon > 0$ we can find step functions h and k such that $h < f < k$ and $\int k - \int h < \epsilon$. Then, let $h(x) = \sum a_i \phi_i$ and $k(x) = \sum b_i \phi_i$, where the ϕ_i are the characteristic functions that are 1 on the rectangle R_i , and 0 outside it, and the R_i are a partition of our domain. Then, $h^2 = \sum_{i,j} a_i a_j \phi_i \phi_j$. However, since $\phi_i(x) \phi_j(x)$ is 0 unless x is in R_i and R_j , all these terms are zero unless $i = j$, in which case they are $a_i^2 \phi_i$.

Therefore, we can calculate $h^2 = \sum a_i^2 \phi_i$ and $k^2 = \sum b_i^2 \phi_i$. Also, since f is bounded, it has maximum M . We can assume WLOG that all $b_i \leq M$, since if some $b_i > M$, we can replace b_i with M while keeping $k(x) > f(x)$ for all $x \in A$. Therefore, $\int (k^2 - h^2) = \sum v(R_i)(b_i - a_i)^2 \leq \sum v(R_i) b_i (b_i - a_i) \leq \sum M v(R_i) (b_i - a_i) = M \int (k - h)$. Therefore, for any $\epsilon > 0$, we can choose step functions h, k with $h < f < k$ and $\int k - \int h < \epsilon/M$ so that h^2, k^2 are step functions with $h^2 < f^2 < k^2$ and $\int k^2 - \int h^2 < \epsilon$. Therefore, f^2 is integrable.

(b) Since f and g are integrable, $f + g$ and $f - g$ are integrable, since integrable functions are a vector space. Therefore, by (a), $(f + g)^2$ and $(f - g)^2$ are integrable. Therefore, $fg = 1/4[(f + g)^2 - (f - g)^2]$ is integrable.

Problem 2 (a) f is clearly bounded, since $|1/q| \leq 1$ for all q . f is integrable, as we have shown on a previous problem set, but it is not continuous at any (x, y) with both x and y rational. Since the set of rationals in $[0, 1] \times [0, 1]$ is not measurable, f is not admissible.

- (b)
1. We showed this on the previous problem set.
 2. If $x \notin \mathbb{Q}$, then $f(x, y) = 0$ for all y , so $\int f(x, y) dy = 0$. If $x \in \mathbb{Q}$, then $f(x, y)$ is simply $1/q$ for rationals p/q and 0 for irrationals, which integrates to 0, which we also showed on a previous problem set.
 3. If $y \in \mathbb{Q}$, then

$$f(x, y) = \begin{cases} 1/q & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

This is not integrable, because $O(f(x, y))$ is not measurable for fixed y , since any set of rectangles containing all points in the graph must have combined volume at least $1/q$, but no rectangle can fit between 0 and the graph of the function.

This does not contradict Fubini's theorem, because Fubini's theorem applies only to situations in which $f_y(x) = f(x, y)$ is integrable for every fixed y , which is clearly not the case here.

Problem 3 R is the region bounded by the circles C_1, C_2, C_3 , and C_4 with equations given by $x^2 + y^2 = 6x$, $x^2 + y^2 = 4x$, $x^2 + y^2 = 8y$, and $x^2 + y^2 = 2y$, respectively. Using the transformations

$$u = \frac{2x}{x^2 + y^2}, v = \frac{2y}{x^2 + y^2}$$

these four equations turn into $u = 1/3, u = 1/2, v = 1/4$, and $v = 1$ respectively, so that the domain of integration is just a rectangle in the (u, v) plane.

Then, using the equations for u and v , we see that $u^2 + v^2 = 4(x^2 + y^2)/(x^2 + y^2)^2 = 4/(x^2 + y^2)$, and our integral

$$\int_R \frac{dx dy}{(x^2 + y^2)^2} = \int_{T(R)} \frac{(u^2 + v^2)^2}{16} dx dy$$

where T is the transformation taking $(x, y) \rightarrow (u, v)$, and all that remains for us to do is to calculate $dx dy$ in terms of $du dv$ and integrate.

$$T'(u, v) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{2(v^2 - u^2)}{(u^2 + v^2)^2} & -\frac{4uv}{(u^2 + v^2)^2} \\ -\frac{4uv}{(u^2 + v^2)^2} & \frac{2(u^2 - v^2)}{(u^2 + v^2)^2} \end{bmatrix}$$

so

$$|T'| = \left| \frac{1}{(u^2 + v^2)^2} [(2v^2 - 2u^2)(2u^2 - 2v^2) - 16u^2v^2] \right| = \frac{4}{(u^2 + v^2)^2}$$

Therefore, our integral becomes

$$\int \frac{(u^2 + v^2)^2}{16} \frac{4}{(u^2 + v^2)^2} du dv = \int \frac{du dv}{4}$$

over a rectangle in the (u, v) plane. This integral is just the area of the rectangle times $1/4$, so the integral comes out to $(1/4)(3/4)(1/6) = 1/32$.

Problem 4 For the following integrals, we simply rewrite the function to be integrated in terms of r and θ , rewrite the limits of integration in terms of r and θ , multiply by a factor of r (since this is the determinant of the change of variable matrix), and then compute.

(a)

$$\int_0^1 \int_x^{x^2} \frac{1}{\sqrt{x^2 + y^2}} dy dx$$

Since $r = \sqrt{x^2 + y^2}$ this is just

$$\int_A \frac{1}{r} dr d\theta = v(A)$$

where A is the area of the domain in the (r, θ) plane. θ goes from 0 to $\pi/4$, and r goes from 0 to the point where $y = x^2$. Since $y = r \sin \theta$ and $x = r \cos \theta$, this point is simply the point where $r = \frac{\sin \theta}{\cos^2 \theta}$. Therefore, we can rewrite this integral as an area integral in the (r, θ) plane, given by

$$\int_0^{\pi/4} \frac{\sin \theta}{\cos^2 \theta} d\theta$$

Since this has antiderivative $\frac{1}{\cos \theta} + C$, we can evaluate this integral to be $\sqrt{2} - 1$.

(b)

$$\begin{aligned} & \int_0^a \int_0^{\sqrt{a^2 - y^2}} (x^2 + y^2) dx dy \\ & \int_0^{\pi/2} \int_0^a r^2(r) dr d\theta = \int_0^{\pi/2} a^4/4 d\theta = \pi a^2/8 \end{aligned}$$

Problem 5

$$\int_{-\infty}^{\infty} e^{-x^2} dx$$

(a) Let $I(r) = \int_{-r}^r e^{-x^2} dx$. Then

$$I(r)^2 = \int_{-r}^r e^{-x^2} dx \int_{-r}^r e^{-y^2} dy = \int_R e^{-(x^2 + y^2)} dx dy$$

where R is the rectangle $[-r, r] \times [-r, r]$.

- (b) Note that $e^{-x^2-y^2}$ is always positive. Since $B(r) \subset R \subset B(2r)$, we then have

$$\int_{B(r)} e^{-x^2-y^2} dx dy \leq I(r)^2 \leq \int_{B(2r)} e^{-x^2-y^2} dx dy$$

- (c) We evaluate this by changing to polar coordinates:

$$\int_{B(r)} e^{-(x^2+y^2)} dx dy = \int_0^{2\pi} \int_0^r r e^{-r^2} dr d\theta = \int_0^{2\pi} \frac{-1}{2} (e^{-r^2} - 1) d\theta = \pi(1 - e^{-r^2})$$

- (d) Then, $\lim_{r \rightarrow \infty} \int_{B(r)} e^{-x^2-y^2} dx dy = \pi$, so by the squeeze theorem

$$\lim_{r \rightarrow \infty} I(r)^2 = \pi$$

Since $I(r)$ is clearly not negative, $\lim_{r \rightarrow \infty} I(r) = \sqrt{\pi}$.

Problem 6 (a)

$$v(B_n(r)) = \int_{B_n(r)} dx_1 dx_2 \dots dx_n$$

If we change variables so that $y_i = x_i/r$, then the vector $x \in B_n(r)$ is equivalent to the vector $y \in B_n$. Since each variable is scaled by a factor of r , the determinant of our change of variable matrix is r^n . Therefore, we have

$$v(B_n(r)) = \int_{B_n} r^n dy_1 dy_2 \dots dy_n = r^n v(B_n)$$

- (b)

$$v(B_1) = \int_{-1}^1 dx = 2$$

$$v(B_2) = \int_0^{2\pi} \int_0^1 r dr d\theta = \int_0^{2\pi} (1/2) d\theta = \pi$$

- (c) If $x_1^2 + \dots + x_n^2 \leq 1 \Leftrightarrow \chi_{B_n} = 1$, then clearly $x_1^2 + x_2^2 \leq 1 \Leftrightarrow \chi_{B_2} = 1$ and $x_3^2 + \dots + x_n^2 \leq 1 - (x_1^2 - x_2^2) \Leftrightarrow \chi_{B_{n-2}(\sqrt{1-x_1^2-x_2^2})} = 1$. If $x_1^2 + \dots + x_n^2 > 1 \Leftrightarrow \chi_{B_n} = 0$, then $x_3^2 + \dots + x_n^2 > 1 - x_1^2 - x_2^2 \Leftrightarrow \chi_{B_{n-2}(\sqrt{1-x_1^2-x_2^2})} = 0$. Since χ only takes the value 0 and 1, this shows that

$$\chi_{B_n} = \chi_{B_2} \chi_{B_{n-2}(\sqrt{1-x_1^2-x_2^2})}$$

(d)

$$\int_{B_2} (1-x_1^2-x_2^2)^{(n-2)}/2 dx_1 dx_2 = \int_0^{2\pi} \int_0^1 r(1-r^2)^{(n-2)}/2 dr d\theta = \int_0^{2\pi} (1/n) d\theta = 2\pi/n$$

(e)

$$\begin{aligned} v(B_n) &= \int \chi_{B_n} dx_1 \dots dx_n = \int \chi_{B_2} \chi_{B_{n-2}(\sqrt{1-x_1^2-x_2^2})} dx_1 \dots dx_n \\ &= \int (\chi_{B_{n-2}(\sqrt{1-x_1^2-x_2^2})} dx_3 \dots dx_n) \chi_{B_2} dx_1 dx_2 = \int_{B_2} (1-x_1^2-x_2^2)^{(n-2)/2} dx_1 dx_2 v(B_{n-2}) \end{aligned}$$

by Fubini. Applying (d) we get that this gives

$$v(B_n) = \frac{2\pi}{n} v(B_{n-2})$$

(f) For odd dimension: The base case holds, since $v(B_1) = 2 = \frac{2\pi^0}{1}$. Assuming that

$$B_{2n+1} = \frac{2^{2n+1} \pi^n}{1 \cdot 3 \cdot \dots \cdot (2n+1)}$$

we have

$$B(2n+3) = 2\pi/(2n+3)v(B_{2n+1}) = \frac{2^{2n+3} \pi^{n+1}}{1 \cdot \dots \cdot (2n+3)}$$

so the formula holds for all odd dimensional spheres.

For even dimension: The base case holds, since $v(B_2) = \pi = \frac{\pi}{0!}$. Assuming that $v(B_{2n}) = \frac{\pi^n}{n!}$ we get $v(B_{2n+2}) = (2\pi)/(2n+2)v(B_{2n}) = \frac{\pi^{n+1}}{(n+1)!}$ and the formula holds for all even dimensional spheres.