PSET 12 SOLUTIONS!!!! LAST ONE!

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1. Problem 1

Note that if $f(x,y,z) = x^2yz$, then $df = 2xyz\ dx + x^2z\ dy + x^2y\ dz$. But then, substituting, the integral is equal to

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)) = 8 - 1 = 7.$$

2. Problem 2

We use the surface area formula

$$A(\sigma) = \int_Q \sqrt{|\sigma'(u)^t\sigma(u)|}.$$

In this case, note that $\sigma(\theta,\phi)$ is a vector with entries $\gamma sin\phi cos\theta$, $\gamma sin\phi sin\theta$ and $\gamma cos\phi$, and the set Q is $[0,2\pi]\times[0,\pi]$. Now we may differentiate and find that $\sigma'(u)$ is a 2 by 3 matrix with rows $(-\gamma sin\phi sin\theta, \gamma cos\phi cos\theta)$, $(\gamma sin\phi cos\theta, \gamma cos\phi sin\theta)$ and $(0, -\gamma sin\phi)$.
Then we compute with matrices to find that

$$\sqrt{|\sigma'(u)^t\sigma'(u)|} = r^2 sin\phi.$$

Then the area is simply

$$\int_Q r^2 sin\phi\ du = 4\pi r^2,$$

by Fubini's theorem.

3. Problem 3

Suppose that the forms are linearly independent. Then we may write $dx_1 \dots dx_k$ as a linear combination of the other elements of the form $dx_{\alpha 1}\dots dx_{\alpha k}$. Just to shorten the typing, write

$$dx_A = dx_{\alpha 1}\dots dx_{\alpha k},$$

and

$$dx_1\dots dx_k = \sum_A dx_A c_A,$$

for some set of constants c_A.
Now evaluate on the subset $e1,\dots ek$ of the standard basis, so that $dx_1\dots dx_k(e1,\dots ek) = 1$. Note that, in this case, each of the other dx_A evaluates to zero since one of the columns in the determinant formula will be all zeroes (the standard basis is LI). But then we have $0 = 1$, which is a contradiction.

4. Problem 4

a) The isomorphism identifies zero forms with smooth functions $\mathbb{R}^n \to \mathbb{R}$. This is because there are no dx in zero forms. this basically follows from definitions. Food for thought: If f is a zero form with $df = 0$, what is special about f?

b)One-forms can be expressed by

$$w = f_1 dx + f_2 dy + f_3 dz.$$

The isomorphism maps w to (f_1, f_2, f_3). Basically, note that this is done mapping dx to the first component vector, dy to the second component vector, and so on.

c) This is the same thing as the previous part, only that

$$w = f_1 dx\ dy + f_2 dy\ dz + f_3 dx\ dz,$$

and we map this to (f_1, f_2, f_3).

d)This is the same thing as before!! :) We map $f_1 dx\ dy\ dz$ to f and viceversa.

e) For a smooth map $f : \mathbb{R}^3 \to \mathbb{R}$, note that

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial z}{\partial z} dz,$$

which maps to $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$, the gradient.

f) If $w = f_1 dx\ dy + f_2 dy\ dz + f_3 dx\ dz$, we differentiate to get

$$dw = (\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}) dx\ dy + (\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}) dx\ dz + (\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}) dy\ dz.$$

This maps to the curl using the isomorphism above.

g) Writing

$$w = f\ dx\ dy + g\ dy\ dz + h\ dx\ dz,$$

we obtain that

$$dw = (\frac{\partial h}{\partial x} - \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}) dx\ dy\ dz.$$

Using a slightly modified isomorphism that will map w to $(f, -g, h)$, we obtain

$$d(f, g, h) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}.$$

That's it! y'all! good luck!
Love,
-Gerardo