MATH 251b: FINAL EXAM

You have 3 hours for this exam. It will be graded out of 100 points. You may attempt as many questions as you like. There are a total of 150 points available, thus you do not need to attempt every question to get full credit. I advise you to think carefully about which questions you will attempt first.

You may not consult any books, notes etc. Nor may you use a calculator. You may use without proof any results we have discussed in the course (including results from the homework) as long as you state them clearly. Otherwise you should fully justify all your answers. GOOD LUCK!

1) [**15 points**] Let K be a p-adic field, let ν be a Haar measure on $K^\times$ and let $\chi : K^\times \longrightarrow \mathbb{C}^\times$ be a continuous character. Find a function $\varphi \in \mathcal{S}(K_v) = C_c^\infty(K_v)$ such that

$$\int_{K^\times} \varphi(x)\chi(x)|x|_K^s d\nu = L(\chi, s).$$

Justify your answer.

2) Let K be a totally real number field (i.e. a number field with no complex place). If $a \in \mathbb{Q}_p$ define $\{a\}_p \in \mathbb{Z}[1/p]/\mathbb{Z}$ by $a - \{a\}_p \in \mathbb{Z}_p$. If v is a place of K define a continuous homomorphism $\psi_v : K_v \to \mathbb{C}^\times$ as follows. If v is real set $\psi_v(x) = e^{2\pi i x}$. If v lies above a rational prime p, set

$$\psi_v(x) = e^{-2\pi i \{\mathrm{tr}\ _{K_v/\mathbb{Q}_p} x\}_p}.$$

Also set

$$\psi = \prod_v \psi_v : \mathbb{A}_K \longrightarrow \mathbb{C}^\times.$$

(a)[**15 points**] Show that

$$\begin{array}{ccc} \mathbb{A}_K & \xrightarrow{\sim} & \widehat{\mathbb{A}}_K \\ a & \longmapsto & \psi \circ a \end{array}$$

is an isomorphism. [If you state them clearly, you may assume general facts about topological groups and local fields.] Show moreover that under this isomorphism

$$K \xrightarrow{\sim} K^\perp.$$

[You may assume that $K \subset \mathbb{A}_K$ is discrete and that $\mathbb{A}_K/K$ is compact.]

(b) [**20 points**] If v is a place of K describe, with justification, a self-dual Haar measure on K_v (with respect to ψ_v).

(c) [**10 points**] Let μ be a Haar measure on $\mathbb{A}_K$ compatible with

$$(0) \longrightarrow K \longrightarrow \mathbb{A}_K \longrightarrow \mathbb{A}_K/K \longrightarrow (0),$$

and the Haar measure on K giving points volume one, and the Haar measure on $\mathbb{A}_K/K$ giving the whole space volume 1. Evaluate

$$\mu \prod_v \{x \in K_v : \ |x|_v \leq 1\}.$$

3) Suppose that L/K is a finite Galois extension of number fields. Suppose also that H is a subgroup of $\mathrm{Gal}\,(L/K)$.

(a) [**6 points**] Show that

$$\#\bigcup_{\sigma \in \mathrm{Gal}\,(L/K)} \sigma H \sigma^{-1} \leq [L:K] + 1 - \#\mathrm{Gal}\,(L/K)/H.$$

(b) [**9 points**] If above all but finitely many primes of K there is a prime of L^H which is split over K, show that $H = \mathrm{Gal}\,(L/K)$.

(c) [**5 points**] If $f(X) \in \mathcal{O}_K[X]$ is irreducible over K and has a root modulo all but finitely many primes of K, show that f has a root in K (i.e. is linear).

4) Suppose that $d \in \mathbb{Z}_{>1}$ is square free and $d \equiv 1 \bmod 4$. Let $p \nmid 2d$ be a rational prime.

(a) [**7 points**] Show that $\mathbb{Z}[\sqrt{-d}]$ is the ring of integers in $\mathbb{Q}(\sqrt{-d})$. Find $\mathbb{Z}[\sqrt{-d}]^\times$.

(b) [**15 points**] Show that $p \in \mathbf{N}_{\mathbb{Q}(\sqrt{-d})/\mathbb{Q}}\mathbb{Q}(\sqrt{-d})^\times$ if and only if

- $p \in N_{\mathbb{Q}_2(\sqrt{-d})/\mathbb{Q}_2}\mathbb{Z}_2[\sqrt{-d}]^\times$;

- and $p \in (\mathbb{Z}_q^\times)^2$ for all primes $q|d$;

- and p splits in $\mathbb{Q}(\sqrt{-d})$.

(c) [**7 points**] Suppose that for all $q|d$ we have $p \in (\mathbb{Z}_q^\times)^2$ and that $p \in N_{\mathbb{Q}_2(\sqrt{-d})/\mathbb{Q}_2}\mathbb{Z}_2[\sqrt{-d}]^\times$. By calculating $r_{\mathbb{Q}(\sqrt{-d})/\mathbb{Q}}(p)$ in two ways, where we consider $p \in \mathbb{Q}^\times \subset \mathbb{A}^\times$, show that p splits in $\mathbb{Q}(\sqrt{-d})$.

(d) [**7 points**] Show that the set of rational primes p of the form $x^2 + dy^2$ with $x, y \in \mathbb{Q}$ has Dirichlet density $1/2^{1+r}$, where r denotes the number of prime factors of d.

(e) [**6 points**] Show that a rational prime $p \nmid 2d$ can be written $p = x^2 + dy^2$ with $x, y \in \mathbb{Z}$ if and only if p splits in $\mathbb{Q}(\sqrt{-d})$ and the primes above p are principal.

(f) [**8 points**] Let h denote the class number of $\mathbb{Q}(\sqrt{-d})$. Show that the set of rational primes of the form $x^2 + dy^2$ with $x, y \in \mathbb{Z}$ has Dirichlet density $1/2h$. [Hint: Consider the Hilbert class field of K.]

5) Let p denote an odd rational prime.

(a) [**3 points**] If $g \in SL_2(\mathbb{F}_p)$ and $g^2 = 1_2$ show that $g = \pm 1_2$. [We write 1_2 for the identity element of $SL_2(\mathbb{F}_p)$.]

(b) [**4 points**] Let $t : SL_2(\mathbb{F}_p)/\{\pm 1_2\} \to SL_2(\mathbb{F}_p)$ be a set theoretic section to the quotient map $SL_2(\mathbb{F}_p) \twoheadrightarrow SL_2(\mathbb{F}_p)/\{\pm 1_2\}$. Suppose that G is a profinite group and that $\bar{r} : G \to SL_2(\mathbb{F}_p)/\{\pm 1_2\}$ is a continuous homomorphism. Recall that the map

$$\begin{aligned} \psi : G \times G &\to \{\pm 1_2\} \\ (\sigma, \tau) &\longmapsto t(\bar{r}(\sigma\tau))t(\bar{r}(\tau))^{-1}t(\bar{r}(\sigma))^{-1} \end{aligned}$$

is a 2-cocycle. Show that $[\psi] \in H^2(G, \{\pm 1_2\})$ is zero if and only if $\bar{r}$ lifts to a homomorphism $r : G \to SL_2(\mathbb{F}_p)$.

(c) [**13 points**] Suppose that

$$\bar{r} : \mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow SL_2(\mathbb{F}_p)/\{\pm 1_2\}$$

is a continuous homomorphism unramified outside $\{\infty, p\}$ (i.e. for any rational prime $q \neq p$, $\bar{r}(I_{\mathbb{Q}_q}) = \{1_2\}$). Show that $\bar{r}$ lifts to a homomorphism

$$r : \mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow SL_2(\mathbb{F}_p)$$

if and only if $\bar{r}(c) = 1$ (where c denotes the non-trivial element of $\mathrm{Gal}\,(\mathbb{C}/\mathbb{R})$).