NUMBER FIELDS

1) (a) Find
$$\{\alpha \in \mathbb{Q}(\sqrt[3]{17}) : \operatorname{tr}_{\mathbb{Q}(\sqrt[3]{17})/\mathbb{Q}} \alpha \mathbb{Z}[\sqrt[3]{17}] \subset \mathbb{Z}\}.$$

(b) Find $\mathcal{O}_{\mathbb{Q}(\sqrt[3]{17})}$.

(c) Find $\mathcal{D}^{-1}_{\mathbb{Q}(\sqrt[3]{17})/\mathbb{Q}}$.

(d) How does (3) factorise in $\mathbb{Q}(\sqrt[3]{17})$?

(e) Let $v$ (resp. $w$) denote the real (resp. complex) place of $\mathbb{Q}(\sqrt[3]{17})$. Find $\alpha \in \mathbb{Q}(\sqrt[3]{17})$ with
$$|\alpha - 1|_v < 1/100$$
and
$$|\alpha + 1|_w < 1/100.$$

2) Suppose that $K$ is either a $p$-adic field or a number field and suppose that $A$ is a central simple finite dimensional $K$ algebra. Thus $A = M_n(D)$ for some division algebra $D$ with centre $K$. Let $L/K$ be a finite extension such that $A \otimes_K L \cong M_n(L)$. By an order $\mathfrak{o} \subset A$ we shall mean a sub-$\mathcal{O}_K$-algebra which spans $A$ over $K$ and which is finitely generated as a $\mathcal{O}_K$-module. We will call an order $\mathfrak{o}$ maximal if it is not properly contained in any other order. Let $\operatorname{tr}_{A/K}$ denote the reduced trace. (See last semester's example sheet.)

(a) Show that $A \cap M_n(\mathcal{O}_L)$ is an order in $A$.

(b) Show that the pairing $A \times A \to K$ given by $(\alpha, \beta) \mapsto \operatorname{tr}_{A/K}(\alpha\beta)$ is perfect. [Hint: Work first over $L$.]

(c) If $\mathfrak{o} \subset A$ is an order, show that all elements of $\mathfrak{o}$ which are invertible in $A$ are integral.

(d) If $\alpha \in A$ show that for some $\beta \in \mathcal{O}_K$ the element $\alpha + \beta$ is invertible in $A$. Deduce that if $\mathfrak{o} \subset A$ is an order then $\operatorname{tr}_{A/K} \mathfrak{o} \subset \mathcal{O}_K$.

(e) Show that if $\mathfrak{o}' \supset \mathfrak{o}$ are two orders in $A$ then
$$\mathfrak{o}' \subset \{\alpha \in A : \operatorname{tr}_{A/K}(\alpha\mathfrak{o}) \subset \mathcal{O}_K\}.$$
Deduce that any order is contained in a maximal order.

(f) Suppose that $K$ is a number field and let $\mathfrak{o} \subset A$ be an order. Show that there is a bijection between the set of all maximal orders $\mathfrak{o}' \subset A$ and the set of sequences $(\mathfrak{o}'_v)_v$, where for each finite place $v$ of $K$, $\mathfrak{o}'_v$ is a maximal order of $A \otimes_K K_v$ with $\mathfrak{o}'_v = \mathfrak{o} \otimes_{\mathcal{O}_K} \mathcal{O}_{K_v}$ for all but finitely many $v$. (The map sends $\mathfrak{o}'$ to $(\mathfrak{o}' \otimes_{\mathcal{O}_K} \mathcal{O}_{K_v})_v$.)

(g) Now suppose that $K$ is a $p$-adic field. By an $\mathcal{O}_D$-lattice in $D^n$ we will mean a right $\mathcal{O}_D$-submodule $M \subset D^n$ which spans $D^n$ over $K$ but which is finitely generated as a $\mathcal{O}_K$-module. Show that any $\mathcal{O}_D$-lattice in $D^n$ is free of rank $n$. [Hint:

Use induction on $n$. Choose $e_1 \in M$ such that the valuation of its first coordinate is minimal. Show that $M$ is the direct sum of $e_1\mathcal{O}_D$ and the submodule of $M$ consisting of elements whose first coordinate is zero.]

(h) Continue to suppose that $K$ is a $p$-adic field. If $\mathfrak{o}$ is any order in $A$ show that it stabilises a $\mathcal{O}_D$-lattice in $D^n$. Deduce that any maximal order in $A$ is conjugate to $M_n(\mathcal{O}_D)$.

(i) Return to the case that $K$ is a number field. Let $\mathfrak{o}$ denote a maximal order in $A$. Write $A_v$ (resp. $\mathfrak{o}_v$) for $A \otimes_K K_v$ (resp. $\mathfrak{o} \otimes_{\mathcal{O}_K} \mathcal{O}_{K_v}$). Show that the set of $A^\times$-conjugacy classes of maximal orders in $A$ is in bijection with

$$A^\times \backslash (\prod_v{}'_{\{\mathfrak{o}_v^\times\}} A_v^\times) / (\prod_v \mathfrak{o}_v^\times)(\prod_v{}'_{\{\mathcal{O}_{K_v}^\times\}} K_v^\times).$$

## ANALYSIS ON TOPOLOGICAL GROUPS

All topological groups are assumed to be Hausdorff, locally compact and second countable, unless they are denoted by $X$ or $Y$.

1) Suppose that $G$ is a topological group and that $\mu$ is a Haar measure on $G$. If $\phi, \psi, \rho$ are elements of $L^1(G,\mu)$ show that $(\phi * \psi) * \rho = \phi * (\psi * \rho)$. Show the same if $\phi, \psi, \rho$ are all elements of $L^2(G,\mu)$.

2) If $X$ and $Y$ are topological spaces let $C(X,Y)$ denote the space of continuous maps from $X$ to $Y$. Make $C(X,Y)$ into a topological space by taking as a basis of open sets all sets of the form:

$$\{f \in C(X,Y) : \; f(A_i) \subset U_i \text{ for } i = 1, ..., n\}$$

where $A_i \subset X$ are a finite collection of compact subsets and where $U_i \subset Y$ are a finite set of open subsets.

(a) If $Y$ is Hausdorff, show that $C(X,Y)$ is Hausdorff.

(b) If $X$ is second countable and locally compact, and if $Y$ is second countable, show that $C(X,Y)$ is second countable.

(c) Show that if $X$ is locally compact and if $Y$ is a topological group then $C(X,Y)$ becomes a topological group if we define $(fg)(x) = f(x)g(x)$. Show moreover that in this case the sets of the form

$$\{f : \; fA \subset U\}$$

as $A$ runs over compact subsets of $X$ and $U$ runs over open neighbourhoods of the identity in $Y$, forms a basis of open neighbourhoods of the identity in $C(X,Y)$.

(d) If $X$ and $Y$ are topological groups and if $Y$ is Hausdorff show that the set of homomorphisms in $C(X,Y)$ is a closed subset.

3) Let $G$ be a topological group. Let $A$ be a compact neighbourhood of 1 in $G$ and let $B$ be a compact neighbourhood of 1 in $S^1$, which is contained in the subset of elements of $S^1$ with positive real part. Let $W(A,B)$ denote the set of elements of $\widehat{G}$, which take $A$ into $B$. Consider the continuous injection $W(A,B) \hookrightarrow (S^1)^G$.

(a) If $\chi$ is in the closure of $W(A,B)$ in $(S^1)^G$ show that $\chi$ is a group homomorphism and that $\chi(A) \subset B$.

(b) Show that for any open neighbourhood $U \subset B$ of the identity in $S^1$ there exists an integer $N = N(U)$ such that if $z \in S^1$ and $z^i \in B$ for $i = 1, ..., N$ then $z \in U$. Let $V$ be a neighbourhood of 1 in $G$ such that $V^N \subset A$. If $\chi$ is in the closure of $W(A,B)$ in $(S^1)^G$ show that $\chi(V) \subset U$. Deduce that $\chi$ is continuous, and hence that the image of $W(A,B)$ is closed in $(S^1)^G$.

(c) Choose $g_1, ..., g_r \in A$ such that $A \subset g_1V \cup ... \cup g_rV$. If $\chi \in W(A,B)$ and $\chi(g_i) \in U$ for $i = 1, ..., r$ show that $\chi A \subset U^2$. Deduce that there is an open set

$U' \subset (S^1)^G$ with $1 \in U' \cap W(A,B) \subset W(A,U^2)$. Conclude that the embedding $W(A,B) \hookrightarrow (S^1)^G$ is a homeomorphism onto its image.

(d) Show that $W(A,B)$ is compact and that $\widehat{G}$ is locally compact.

[In class someone suggested this local compactness was a consequence of some more general phenomenon. If this is so, could someone educate me about this?]

4) (a) If $G$ is a discrete topological group show that $\widehat{G}$ is compact.

(b) If $G$ is a compact topological group show that $\widehat{G}$ is discrete.

5) Let $K$ be a $p$-adic field and $\chi : K^\times \to \mathbb{C}^\times$ a continuous homomorphism.

(a) Show that there is a unique (up to scalar multiples) $K^\times$-equivariant $\mathbb{C}$-linear map $C_c^\infty(K^\times) \to \mathbb{C}(\chi)$. [Hint: Suppose that $\chi|_{1+\wp_K^f} = 1$ for some $f \in \mathbb{Z}_{>0}$. Show that any such linear map is uniquely determined by the image of the characteristic function of $1 + \wp_K^f$.]

(b) Show that there is an exact sequence

$$(0) \to C_c^\infty(K^\times) \to \mathcal{S}(K) \to \mathbb{C} \to (0)$$

where the third map is evaluation at 0. Deduce that if $\chi \neq 1$ then there is at most one (up to scalar multiples) $K^\times$-equivariant $\mathbb{C}$-linear map $\mathcal{S}(K) \to \mathbb{C}(\chi)$. Show that the same conclusion holds even for $\chi = 1$. [Hint: For the last assertion, note that the characteristic functions of $\mathcal{O}_K$ and $\wp_K$ must have the same image. Deduce that the characteristic function of $1 + \wp_K$ must map to zero.]

(c) Use the results of part (b) to give another proof of the local functional equation.

(d) Show that for all $\chi$ there is a non-trivial $K^\times$-equivariant $\mathbb{C}$-linear map $\mathcal{S}(K) \to \mathbb{C}(\chi)$.

6) Suppose that $\varphi \in \mathcal{S}(\mathbb{C})$, that

$$\chi(x) = (x/|x|_{\mathbb{C}}^{1/2})^n |x|_{\mathbb{C}}^t$$

is a continuous homomorphism ($n \in \mathbb{Z}$ and $t \in \mathbb{C}$) and that $\nu$ is a Haar measure on $\mathbb{C}^\times$.

(a) Show that $\zeta(\varphi, \chi, \nu, s)$ converges for $\mathrm{Re}\, s >> 0$, that it has meromorphic continuation to the whole complex plane and that $\zeta(\varphi, \chi, \nu, s)/L(\chi, s)$ is entire.

(b) Show that for some choice of $\varphi$ (depending on $\chi$) we have $L(\chi, s) = \zeta(\varphi, \chi, \nu, s)$. Also show that $\epsilon(\chi, 2\lambda, \exp(4\pi i \mathrm{Re}\,), s) = i^{|n|}$, where $\lambda$ denotes the usual Lebesgue measure.

7) Let $K$ denote a $p$-adic field. Let $\psi$ denote a non-trivial element of $\widehat{K}$, let $\mu$ denote a Haar measure on $K$ and let $\ker\psi = \wp_K^{-d}$. Further let $\chi : K^\times \to \mathbb{C}^\times$ be a continuous homomorphism with conductor $f$ (i.e. $f = 0$ if and only if $\chi|_{\mathcal{O}_K^\times} = 1$, and otherwise $f$ is the smallest positive integer such that $\chi|_{1+\wp_K^f} = 1$). Suppose that $a \in \mathbb{Z}_{\geq 0}$ with $2a \leq f$.

(a) Show that there is an element $\alpha \in \wp_K^{-f-d} - \wp_K^{1-f-d}$ such that for $x \in \wp_K^{f-a}$ we have

$$\chi(1+x) = \psi(\alpha x).$$

Show moreover that $\alpha$ is unique modulo $\wp_K^{a-f-d}$.

(b) Show that with this choice of $\alpha$ we have

$$\epsilon(\chi,\psi,\mu,s) = |\varpi_K|_K^{(d+f)s}\mu(\wp_K^{-a-d})\chi(-1)\sum_{x\in(1+\wp_K^a)/(1+\wp_K^{f-a})}\psi(\alpha x)\chi(\alpha x)^{-1}.$$

(If $a = 0$ we interpret $1 + \wp_K^a$ as $\mathcal{O}_K^\times$.)

(c) Suppose that $\chi' : K^\times \to \mathbb{C}^\times$ is a continuous homomorphism with conductor $f'$. If $2f' \leq f$ show that

$$\epsilon(\chi\chi',\psi,\mu,s) = \chi'(-\alpha)^{-1}\epsilon(\chi,\psi,\mu,s).$$

(d) Suppose that $L/K$ is a finite extension with ramification index $e$. Assume further that if $a = 0$ then $e = 1$. Let $\mu'$ denote a Haar measure on $L$.

Show that if $a > 1$ and $\wp_K^{f-2a} \subset \mathcal{D}_{L/K}^2$ then for $x \in \mathcal{D}_{L/K}^{-1}\wp_K^{f-a}$ one has

$$\chi(N_{L/K}(1+x)) = \psi(\mathrm{tr}\,_{L/K}(\alpha y)).$$

Deduce that in this case $\chi \circ N_{L/K}$ has conductor $ef - v_L(\mathcal{D}_{L/K})$ and $\alpha\mathcal{O}_L = \mathcal{D}_{L/\mathbb{Q}_p}^{-1}(\mathcal{D}_{L/K}\wp_K^{-f})$.

Suppose that $\chi' : L^\times \to \mathbb{C}^\times$ is a continuous homomorphism with conductor $f'$. If $f' \leq ae$ and $2ae \leq ef - 2v_L(\mathcal{D}_{L/K})$ show that

$$\epsilon((\chi\circ N_{L/K})\chi',\psi,\mu,s) = \chi'(-\alpha)^{-1}\epsilon(\chi\circ N_{L/K},\psi,\mu,s).$$

[Hint: Treat the cases $a > 1$ and $a = 0$, $e = 1$ separately.]

(e) Suppose that $\mu_\psi^* = \mu$. Show that

$$\chi(-\alpha)\epsilon(\chi,\psi,\mu,1/2) = |\varpi_K|_K^{f/2-a}\sum_{x\in(1+\wp_K^a)/(1+\wp_K^{f-a})}\psi(\alpha x)\chi(x)^{-1}$$

and that this lies in some number field $E$. If $v$ is a finite place of $E$ not dividing $p$ show that $\chi(-\alpha)\epsilon(\chi,\psi,\mu,1/2)$ is integral at $v$. Deduce that in fact it is a unit at $v$. [Hint: Consider the product $\epsilon(\chi,\psi,\mu,1/2)\epsilon(\chi^{-1},\psi^{-1},\mu,1/2)$.]

Now suppose further that $|\chi| = 1$. Show that

$$|\chi(-\alpha)\epsilon(\chi, \psi, \mu, 1/2)|_v = 1$$

for each infinite place $v$ of $E$. If in addition $f \neq 1$ show that

$$(\chi(-\alpha)\epsilon(\chi, \psi, \mu, 1/2))^2$$

lies in $\mathbb{Q}(\zeta_{p^n})$ for some $n$. Use the product formula to deduce that in this case ($\mu_\psi^* = \mu$, $|\chi| = 1$ and $f \neq 1$) $\chi(-\alpha)\epsilon(\chi, \psi, \mu, 1/2)$ is a root of unity.

8) (a) Suppose that $G$ is a profinite group, that $H$ is an open subgroup and that $\chi : H \to \mathbb{C}^\times$ is a continuous homomorphism. Show that

$$\det \mathrm{Ind}\,_H^G \chi = \mathrm{cor}\chi \in H^1(G, \mathbb{C}^\times).$$

(b) Now suppose that $K$ is a $p$-adic field and that $(\rho, V)$ is a representation, with open kernel, of $\mathrm{Gal}\,(\overline{K}/K)$ on a finite dimensional $\mathbb{C}$-vector space $V$. Recall that we can find finite extensions $L_i/K$, characters $\phi_i : \mathrm{Gal}\,(\overline{K}/L_i) \to C^\times$ with open kernels, and integers $n_i \in \mathbb{Z}$ such that

$$\rho - (\dim \rho)1 = \sum_i n_i \mathrm{Ind}\,_{\mathrm{Gal}\,(\overline{K}/L_i)}^{\mathrm{Gal}\,(\overline{K}/K)}(\phi_i - 1).$$

Show that

$$\det \rho = \prod_i (\mathrm{cor}\phi_i)^{n_i}.$$

Also show that there is a positive integer $f$ such that for any character $\chi : \mathrm{Gal}\,(\overline{K}/K) \to \mathbb{C}^\times$ with open kernel and conductor $\geq f$ there is an element $\sigma_\chi \in \mathrm{Gal}\,(\overline{K}/K)$ such that

$$\epsilon(\rho \otimes \chi, \psi, \mu, s) = (\det \rho)(\sigma_\chi)\epsilon(\chi, \psi, \mu, s)^{\dim \rho}.$$