

ANALYSIS ON TOPOLOGICAL GROUPS

All topological groups are assumed to be Hausdorff, locally compact and second countable, unless they are denoted by X or Y .

1) Suppose that G is a topological group and that μ is a Haar measure on G . If ϕ, ψ, ρ are elements of $L^1(G, \mu)$ show that $(\phi * \psi) * \rho = \phi * (\psi * \rho)$. Show the same if ϕ, ψ, ρ are all elements of $L^2(G, \mu)$.

2) If X and Y are topological spaces let $C(X, Y)$ denote the space of continuous maps from X to Y . Make $C(X, Y)$ into a topological space by taking as a basis of open sets all sets of the form:

$$\{f \in C(X, Y) : f(A_i) \subset U_i \text{ for } i = 1, \dots, n\}$$

where $A_i \subset X$ are a finite collection of compact subsets and where $U_i \subset Y$ are a finite set of open subsets.

(a) If Y is Hausdorff, show that $C(X, Y)$ is Hausdorff.

(b) If X is second countable and locally compact, and if Y is second countable, show that $C(X, Y)$ is second countable.

(c) Show that if X is locally compact and if Y is a topological group then $C(X, Y)$ becomes a topological group if we define $(fg)(x) = f(x)g(x)$. Show moreover that in this case the sets of the form

$$\{f : fA \subset U\}$$

as A runs over compact subsets of X and U runs over open neighbourhoods of the identity in Y , forms a basis of open neighbourhoods of the identity in $C(X, Y)$.

(d) If X and Y are topological groups and if Y is Hausdorff show that the set of homomorphisms in $C(X, Y)$ is a closed subset.

3) Let G be a topological group. Let A be a compact neighbourhood of 1 in G and let B be a compact neighbourhood of 1 in S^1 , which is contained in the subset of elements of S^1 with positive real part. Let $W(A, B)$ denote the set of elements of \widehat{G} , which take A into B . Consider the continuous injection $W(A, B) \hookrightarrow (S^1)^G$.

(a) If χ is in the closure of $W(A, B)$ in $(S^1)^G$ show that χ is a group homomorphism and that $\chi(A) \subset B$.

(b) Show that for any open neighbourhood $U \subset B$ of the identity in S^1 there exists an integer $N = N(U)$ such that if $z \in S^1$ and $z^i \in B$ for $i = 1, \dots, N$ then $z \in U$. Let V be a neighbourhood of 1 in G such that $V^N \subset A$. If χ is in the closure of $W(A, B)$ in $(S^1)^G$ show that $\chi(V) \subset U$. Deduce that χ is continuous, and hence that the image of $W(A, B)$ is closed in $(S^1)^G$.

(c) Choose $g_1, \dots, g_r \in A$ such that $A \subset g_1V \cup \dots \cup g_rV$. If $\chi \in W(A, B)$ and $\chi(g_i) \in U$ for $i = 1, \dots, r$ show that $\chi A \subset U^2$. Deduce that there is an open set

$U' \subset (S^1)^G$ with $1 \in U' \cap W(A, B) \subset W(A, U^2)$. Conclude that the embedding $W(A, B) \hookrightarrow (S^1)^G$ is a homeomorphism onto its image.

(d) Show that $W(A, B)$ is compact and that \widehat{G} is locally compact.

[In class someone suggested this local compactness was a consequence of some more general phenomenon. If this is so, could someone educate me about this?]

4) (a) If G is a discrete topological group show that \widehat{G} is compact.

(b) If G is a compact topological group show that \widehat{G} is discrete.

5) Let K be a p -adic field and $\chi : K^\times \rightarrow \mathbb{C}^\times$ a continuous homomorphism.

(a) Show that there is a unique (up to scalar multiples) K^\times -equivariant \mathbb{C} -linear map $C_c^\infty(K^\times) \rightarrow \mathbb{C}(\chi)$. [Hint: Suppose that $\chi|_{1+\wp_K^f} = 1$ for some $f \in \mathbb{Z}_{>0}$. Show that any such linear map is uniquely determined by the image of the characteristic function of $1 + \wp_K^f$.]

(b) Show that there is an exact sequence

$$(0) \rightarrow C_c^\infty(K^\times) \rightarrow \mathcal{S}(K) \rightarrow \mathbb{C} \rightarrow (0)$$

where the third map is evaluation at 0. Deduce that if $\chi \neq 1$ then there is at most one (up to scalar multiples) K^\times -equivariant \mathbb{C} -linear map $\mathcal{S}(K) \rightarrow \mathbb{C}(\chi)$. Show that the same conclusion holds even for $\chi = 1$. [Hint: For the last assertion, note that the characteristic functions of \mathcal{O}_K and \wp_K must have the same image. Deduce that the characteristic function of $1 + \wp_K$ must map to zero.]

(c) Use the results of part (b) to give another proof of the local functional equation.

(d) Show that for all χ there is a non-trivial K^\times -equivariant \mathbb{C} -linear map $\mathcal{S}(K) \rightarrow \mathbb{C}(\chi)$.

6) Suppose that $\varphi \in \mathcal{S}(\mathbb{C})$, that

$$\chi(x) = (x/|x|_{\mathbb{C}}^{1/2})^n |x|_{\mathbb{C}}^t$$

is a continuous homomorphism ($n \in \mathbb{Z}$ and $t \in \mathbb{C}$) and that ν is a Haar measure on \mathbb{C}^\times .

(a) Show that $\zeta(\varphi, \chi, \nu, s)$ converges for $\operatorname{Re} s \gg 0$, that it has meromorphic continuation to the whole complex plane and that $\zeta(\varphi, \chi, \nu, s)/L(\chi, s)$ is entire.

(b) Show that for some choice of φ (depending on χ) we have $L(\chi, s) = \zeta(\varphi, \chi, \nu, s)$. Also show that $\epsilon(\chi, 2\lambda, \exp(4\pi i \operatorname{Re}), s) = i^{|n|}$, where λ denotes the usual Lebesgue measure.

7) Let K denote a p -adic field. Let ψ denote a non-trivial element of \widehat{K} , let μ denote a Haar measure on K and let $\ker \psi = \wp_K^{-d}$. Further let $\chi : K^\times \rightarrow \mathbb{C}^\times$ be a continuous homomorphism with conductor f (i.e. $f = 0$ if and only if $\chi|_{\mathcal{O}_K^\times} = 1$, and otherwise f is the smallest positive integer such that $\chi|_{1+\wp_K^f} = 1$). Suppose that $a \in \mathbb{Z}_{\geq 0}$ with $2a \leq f$.

(a) Show that there is an element $\alpha \in \wp_K^{-f-d} - \wp_K^{1-f-d}$ such that for $x \in \wp_K^{f-a}$ we have

$$\chi(1+x) = \psi(\alpha x).$$

Show moreover that α is unique modulo \wp_K^{a-f-d} .

(b) Show that with this choice of α we have

$$\epsilon(\chi, \psi, \mu, s) = |\varpi_K|_K^{(d+f)s} \mu(\wp_K^{-a-d}) \chi(-1) \sum_{x \in (1+\wp_K^a)/(1+\wp_K^{f-a})} \psi(\alpha x) \chi(\alpha x)^{-1}.$$

(If $a = 0$ we interpret $1 + \wp_K^a$ as \mathcal{O}_K^\times .)

(c) Suppose that $\chi' : K^\times \rightarrow \mathbb{C}^\times$ is a continuous homomorphism with conductor f' . If $2f' \leq f$ show that

$$\epsilon(\chi\chi', \psi, \mu, s) = \chi'(-\alpha)^{-1} \epsilon(\chi, \psi, \mu, s).$$

(d) Suppose that L/K is a finite extension with ramification index e . Assume further that if $a = 0$ then $e = 1$. Let μ' denote a Haar measure on L .

Show that if $a > 1$ and $\wp_K^{f-2a} \subset \mathcal{D}_{L/K}^2$ then for $x \in \mathcal{D}_{L/K}^{-1} \wp_K^{f-a}$ one has

$$\chi(N_{L/K}(1+x)) = \psi(\text{tr}_{L/K}(\alpha y)).$$

Deduce that in this case $\chi \circ N_{L/K}$ has conductor $ef - v_L(\mathcal{D}_{L/K})$ and $\alpha \mathcal{O}_L = \mathcal{D}_{L/\mathbb{Q}_p}^{-1}(\mathcal{D}_{L/K} \wp_K^{-f})$.

Suppose that $\chi' : L^\times \rightarrow \mathbb{C}^\times$ is a continuous homomorphism with conductor f' . If $f' \leq ae$ and $2ae \leq ef - 2v_L(\mathcal{D}_{L/K})$ show that

$$\epsilon((\chi \circ N_{L/K})\chi', \psi, \mu, s) = \chi'(-\alpha)^{-1} \epsilon(\chi \circ N_{L/K}, \psi, \mu, s).$$

[Hint: Treat the cases $a > 1$ and $a = 0, e = 1$ separately.]

(e) Suppose that $\mu_\psi^* = \mu$. Show that

$$\chi(-\alpha) \epsilon(\chi, \psi, \mu, 1/2) = |\varpi_K|_K^{f/2-a} \sum_{x \in (1+\wp_K^a)/(1+\wp_K^{f-a})} \psi(\alpha x) \chi(x)^{-1}$$

and that this lies in some number field E . If v is a finite place of E not dividing p show that $\chi(-\alpha) \epsilon(\chi, \psi, \mu, 1/2)$ is integral at v . Deduce that in fact it is a unit at v .

[Hint: Consider the product $\epsilon(\chi, \psi, \mu, 1/2) \epsilon(\chi^{-1}, \psi^{-1}, \mu, 1/2)$.]

Now suppose further that $|\chi| = 1$. Show that

$$|\chi(-\alpha)\epsilon(\chi, \psi, \mu, 1/2)|_v = 1$$

for each infinite place v of E . If in addition $f \neq 1$ show that

$$(\chi(-\alpha)\epsilon(\chi, \psi, \mu, 1/2))^2$$

lies in $\mathbb{Q}(\zeta_{p^n})$ for some n . Use the product formula to deduce that in this case ($\mu_\psi^* = \mu$, $|\chi| = 1$ and $f \neq 1$) $\chi(-\alpha)\epsilon(\chi, \psi, \mu, 1/2)$ is a root of unity.

8) (a) Suppose that G is a profinite group, that H is an open subgroup and that $\chi : H \rightarrow \mathbb{C}^\times$ is a continuous homomorphism. Show that

$$\det \text{Ind}_H^G \chi = \text{cor} \chi \in H^1(G, \mathbb{C}^\times).$$

(b) Now suppose that K is a p -adic field and that (ρ, V) is a representation, with open kernel, of $\text{Gal}(\overline{K}/K)$ on a finite dimensional \mathbb{C} -vector space V . Recall that we can find finite extensions L_i/K , characters $\phi_i : \text{Gal}(\overline{K}/L_i) \rightarrow \mathbb{C}^\times$ with open kernels, and integers $n_i \in \mathbb{Z}$ such that

$$\rho - (\dim \rho)1 = \sum_i n_i \text{Ind}_{\text{Gal}(\overline{K}/L_i)}^{\text{Gal}(\overline{K}/K)} (\phi_i - 1).$$

Show that

$$\det \rho = \prod_i (\text{cor} \phi_i)^{n_i}.$$

Also show that there is a positive integer f such that for any character $\chi : \text{Gal}(\overline{K}/K) \rightarrow \mathbb{C}^\times$ with open kernel and conductor $\geq f$ there is an element $\sigma_\chi \in \text{Gal}(\overline{K}/K)$ such that

$$\epsilon(\rho \otimes \chi, \psi, \mu, s) = (\det \rho)(\sigma_\chi) \epsilon(\chi, \psi, \mu, s)^{\dim \rho}.$$