

ADELES

1) Let K be a number field.

(a) Show that $\mathbb{A}_K^\times = (\lim_{\leftarrow I} \mathcal{O}_K/I) \otimes_{\mathcal{O}_K} K$.

(b) Show that $\mathbb{A}_K = \mathbb{A} \otimes_{\mathbb{Q}} K$.

(c) Show that \mathbb{A}^\times is not closed in \mathbb{A} and that the topology on \mathbb{A}^\times is not the subspace topology for $\mathbb{A}^\times \subset \mathbb{A}$.

2) Let K be a number field. Write N (resp. r_1 , resp. r_2) for $[K : \mathbb{Q}]$ (resp. the number of real places of K , resp. the number of complex places of K).

(a) If I is a non-zero ideal of \mathcal{O}_K , show that there is a non-zero element $\alpha \in I$ with

$$|\mathbf{N}_{K/\mathbb{Q}}\alpha| \leq N!/N^N(4/\pi)^{r_2}[\mathcal{O}_K : I][\mathcal{D}_{K/\mathbb{Q}}^{-1} : \mathcal{O}_K]^{1/2}.$$

(This is due to Minkowski.) [Hint: Let \hat{I} denote the closure of I in \mathbb{A}^\times . Consider the set

$$\Omega = \hat{I} \times \{x \in K_\infty : \sum_{v|\infty} [K_v : \mathbb{R}] |x_v|_v \leq C\} \subset \mathbb{A}_K.$$

Calculate $\mu(\Omega)$. Also show that if $x, y \in \Omega$ and $\alpha = x - y \in K^\times$ then

$$|\mathbf{N}_{K/\mathbb{Q}}\alpha| \leq (2C/N)^N.]$$

(b) Show that any ideal class of \mathcal{O}_K contains an ideal I with

$$[\mathcal{O}_K : I] \leq N!/N^N(4/\pi)^{r_2}[\mathcal{D}_{K/\mathbb{Q}}^{-1} : \mathcal{O}_K]^{1/2}.$$

(c) Find the class group of $\mathbb{Q}(\sqrt[3]{17})$.

(d) Show that for $N > 1$ we have

$$[\mathcal{D}_{K/\mathbb{Q}}^{-1} : \mathcal{O}_K]^{1/2} \geq N^N/N!(\pi/4)^N > (\pi^2/8)(1.66)^{N-2}.$$

(e) Show that if K/\mathbb{Q} is unramified at all finite places then $K = \mathbb{Q}$.

(f) Show that $[K : \mathbb{Q}]$ can be bounded in terms of $[\mathcal{D}_{K/\mathbb{Q}}^{-1} : \mathcal{O}_K]$. [Parts (e) and (f) are special to \mathbb{Q} . They do not remain true for any base number field.]

(g) Show that for any $d \in \mathbb{R}_{>0}$ there are only finitely many number fields (an any given algebraic closure of \mathbb{Q}) with $[\mathcal{D}_{K/\mathbb{Q}}^{-1} : \mathcal{O}_K] \leq d$.

3) Suppose that $\chi : \mathbb{A}_K^\times/K^\times \rightarrow \mathbb{C}^\times$ is a continuous homomorphism. Suppose also that ν is a Haar measure on \mathbb{A}_K^\times and that μ is the canonical Haar measure on \mathbb{A}_K . Let $\varphi \in \mathcal{S}(\mathbb{A}_K)$. Finally let ψ be a non-trivial character of \mathbb{A}_K/K

(a) Show that $\mathcal{F}_{\psi, \mu}(\phi)(0)$ is independent of the choice of ψ .

(b) Show that $\zeta(\mathcal{F}_{\psi, \mu}(\varphi), \chi, \nu, s)$ is independent of the choice of ψ .

(c) Show that $\epsilon(\chi, s) = \prod_v \epsilon(\chi_v, \psi_v, \mu_v, s)$ is independent of the choice of ψ .

4) Let K be a number field with r infinite places and containing w roots of unity. Let h denote the order of the class group of \mathcal{O}_K and let R denote the volume of

$$\ker(\mathbb{R}^I \xrightarrow{\sum} \mathbb{R}) / \log_{\infty} \mathcal{O}_K^{\times}.$$

(We use standard Lebesgue measure on \mathbb{R}^d and on \mathbb{R} and the measure on $\log_{\infty} \mathcal{O}_K^{\times}$ giving each point volume one.) Show that the residue at $s = 0$ of $L(1, s)$ is

$$-2^r hR/w.$$

Find the residue of $L(1, s)$ at $s = 1$.

5) Let N be a positive integer and set $L = \mathbb{Q}(\zeta_N)$, where ζ_N is a primitive N^{th} root of 1.

(a) Show that if v is a place of L above a rational prime p then $\mathbf{N}_{L_v/\mathbb{Q}_p} \mathcal{O}_{L,v}^{\times}$ equals \mathbb{Z}_p^{\times} if $p \nmid N$ and equals $1 + N\mathbb{Z}_p$ if $p \mid N$.

(b) Show that $\mathbb{A}^{\times} = \mathbb{Q}^{\times} \times \mathbb{R}_{>0}^{\times} \times \prod_p \mathbb{Z}_p^{\times}$. Deduce that

$$\mathbb{A}^{\times} / \mathbb{Q}^{\times} \mathbf{N}_{L/\mathbb{Q}} \mathbb{A}_L^{\times} = \prod_p \mathbb{Z}_p^{\times} / \mathbf{N}_{L/\mathbb{Q}} \prod_v \mathcal{O}_{L,v}^{\times} \cong (\mathbb{Z}/N\mathbb{Z})^{\times},$$

where the product is over all finite places v of L . Show moreover that if $p \nmid N$ then $p \in \mathbb{Q}_p^{\times}$ maps to p^{-1} in $(\mathbb{Z}/N\mathbb{Z})^{\times}$.

(c) If $a \in (\mathbb{Z}/N\mathbb{Z})^{\times}$ show that the Dirichlet density of the set of primes $p \equiv a \pmod{N}$ is $1/\#(\mathbb{Z}/N\mathbb{Z})^{\times}$. (This is Dirichlet's theorem on primes in an arithmetic progression.)