FINAL EXAM: SOLUTIONS

MATH 25

1. Simultaneous Diagonalisation

Let f and g be two endomorphisms of a d-dimensional vector space V over a field K. We say that f and g are simultaneously diagonalisable *if there exists a basis $e_1, \cdots, e_d$ of V and elements $\{\lambda_i\}_{i=1\cdots d}$ and $\{\mu_i\}_{i=1\cdots d}$ of K such that $f(e_i) = \lambda_i e_i$ and $g(e_i) = \mu_i e_i$ for $i = 1 \cdots d$.*

Observe that if f and g are simultaneously diagonalisable, then $fg = gf$. In the rest of this problem, we will assume that f and g are diagonalisable and that $fg = gf$. The goal of the problem is to prove that this implies that f and g are simultaneously diagonalisable.

1.1. *Prove that if W is a sub-vector space of V, stable under f, then the restriction of f to W is diagonalisable.*

Since f is diagonalisable, there exists a polynomial $P \in K[X]$ with simple roots such that $P(f) = 0$ on V. Since W is a subspace of V, $P(f) = 0$ a fortiori on W and this shows that the restriction of f to W is diagonalisable.

1.2. *Prove that if $\mu \in K$, then the vector space $\ker(g - \mu\,\mathrm{Id})$ is stable under f.*

We need to show that if $g(x) = \mu x$ then $g(f(x)) = \mu f(x)$ but we know that $g(f(x)) = f(g(x))$ and therefore $g(f(x)) = f(g(x)) = f(\mu x) = \mu f(x)$.

1.3. *Prove that f and g are simultaneously diagonalisable.*

Since g is diagonalisable, we can write $V = \oplus V_\mu$ where $V_\mu = \ker(g - \mu\,\mathrm{Id})$ and μ runs over the eigenvalues of g. We know by 1.2 that each V_μ is stable under f and we know by 1.1 that f is diagonalisable on each V_μ. Each V_μ therefore has a basis consisting of eigenvectors of f and they are also eigenvectors of g since any element of V_μ is one.

The union of those bases over μ gives a basis of V consisting of vectors which are eigenvectors for both f and g so that f and g are simultaneously diagonalisable.

2. Topology of some spaces of matrices

In the remainder of this exam we will study the topology of certain spaces of matrices with complex coefficients. The absolute value $|\cdot|$ is the usual one on $\mathbf{C}$. We will need to use three topological spaces:

- *The space $\mathbf{C}^n$ is the space of n-uples $(x_1, \cdots, x_n)$ with the distance defined by $d(x, y) = \sup_{1 \le i \le n} |x_i - y_i|$ if $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$.*
- *The space $\mathbf{C}[X]_n$ is the set of polynomials $P(X) = X^n + p_{n-1}X^{n-1} + \cdots + p_0$ with the distance defined by $d(P, Q) = \sup_{0 \le i \le n-1} |p_i - q_i|$ if $P(X) = X^n + p_{n-1}X^{n-1} + \cdots + p_0$ and $Q(X) = X^n + q_{n-1}X^{n-1} + \cdots + q_0$.*

- *The space* $\mathrm{M}(n, \mathbf{C})$ *is the space of all* $n \times n$ *matrices with complex coefficients. If* $A = (a_{ij})_{1 \le i,j \le n}$ *and* $B = (b_{ij})_{1 \le i,j \le n}$*, we define* $d(A, B) = \sup_{1 \le i,j \le n} |a_{ij} - b_{ij}|$.

2.1. *Explain* briefly *why the following maps are continuous. You should not use* ϵ/δ *arguments; no more than a few lines of justification are necessary:*

1. $\det : \mathrm{M}(n, \mathbf{C}) \to \mathbf{C}$.
2. *the "characteristic polynomial" map* $\chi : \mathrm{M}(n, \mathbf{C}) \to \mathbf{C}[X]_n$ *which maps* M *to* $\chi_M = \det(X \,\mathrm{Id} - M)$.
3. *the map* $f : \mathbf{C}^n \to \mathbf{C}[X]_n$ *which maps* $(z_1, \cdots, z_n)$ *to* $\prod_{i=1}^n (X - z_i)$.
4. *the "conjugation by* P*" map* $M \mapsto PMP^{-1}$ *from* $\mathrm{M}(n, \mathbf{C})$ *to itself, where* P *is an invertible matrix.*

All those maps are basically given by polynomials in the variables which occur and are therefore continuous.

2.2. *Prove that the set of invertible matrices* $M \in \mathrm{M}(n, \mathbf{C})$ *is open in* $\mathrm{M}(n, \mathbf{C})$.

We've seen in class that M is invertible if and only if $\det(M) \neq 0$. This means that the set of invertible matrices in $\mathrm{M}(n, \mathbf{C})$ is $\det^{-1}(\mathbf{C}^*)$ which is open in $\mathrm{M}(n, \mathbf{C})$ because $\mathbf{C}^*$ is open in $\mathbf{C}$ and det is a continuous map by 2.1.

3. Diagonalisable matrices

Let $\mathrm{D}(n, \mathbf{C})$ *be the subset of* $\mathrm{M}(n, \mathbf{C})$ *consisting of diagonalisable matrices. We will show that* $\mathrm{D}(n, \mathbf{C})$ *is dense in* $\mathrm{M}(n, \mathbf{C})$ *and we will compute its interior.*

3.1. *Let* M *be a matrix which is upper triangular. Prove that each diagonal entry is an eigenvalue of* M.

If the diagonal entries are the m_{ii}'s then $\det(X \,\mathrm{Id} - M) = \prod_i (X - m_{ii})$ and we've seen in class that the eigenvalues are exactly the roots of the characteristic polynomial, so they are exactly the m_{ii}'s.

3.2. *Prove that* $\mathrm{D}(n, \mathbf{C})$ *is dense in* $\mathrm{M}(n, \mathbf{C})$.

Let M be a matrix in $\mathrm{M}(n, \mathbf{C})$. We know that there exists an invertible matrix P such that $Q = P^{-1}MP$ is upper triangular. Let Q_i be a matrix which is the same as Q except that the coefficients $q_1, \cdots, q_n$ on the diagonal are changed to some complex numbers $q_1^i, \cdots, q_n^i$ such that: (1) for $j = 1 \cdots n$, $|q_j^i - q_j| < 1/i$ and (2) the numbers $q_1^i, \cdots, q_n^i$ are pairwise distinct.

Condition (1) shows that $Q_i \to Q$. Condition (2) shows that Q_i has n distinct eigenvalues (by 2.1), and therefore it is diagonalisable. Since the map $X \mapsto PXP^{-1}$ is continuous by 2.1, $Q_i \to Q$ implies that $PQ_iP^{-1} \to M$ and each PQ_iP^{-1} is in $\mathrm{D}(n, \mathbf{C})$. Therefore, $\mathrm{D}(n, \mathbf{C})$ is dense in $\mathrm{M}(n, \mathbf{C})$.

3.3. *Prove that if* E *is a non-empty subset of* $\mathbf{C}^n$ *then* E *is bounded if and only if* $f(E) \subset \mathbf{C}[X]_n$ *is bounded (here* f *is the map from (3) of exercise 2.1).*

It is clear that if E is bounded, then $f(E)$ is bounded - indeed, if $|z_i| < M$ and $P(X) = \prod (X - z_i) = X^n + p_{n-1}X^{n-1} + \cdots + p_0$, then p_j is the sum of all possible products of $n - j$ distinct z_i's and so $|p_j| < M^{n-j}\binom{n}{j}$.

The reverse implication is your homework number 9, problem A.3. It says that if you bound the coefficients of P, then you can bound its roots, so that $f(E)$ bounded implies E bounded.

3.4. *Use this to prove that the set of polynomials having only simple roots is open in* $\mathbf{C}[X]_n$.

This is the same as proving that the set of polynomials having a double (or worse) root is closed. Let the P_i's be polynomials having a double root z_i and such that $P_i \to P$ where P is a polynomial. We will prove that P also has a double root.

Since $P_i \to P$, there is some closed and bounded ball E of $\mathbf{C}[X]_n$ containing the P_i's and P. By 3.3, $f(E)$ is bounded so $\overline{f(E)}$ is compact, and some subsequence $z_{\varphi(i)}$ of the z_i's will have a limit z.

Since $z_{\varphi(i)}$ is a double root of $P_{\varphi(i)}$ and $P_{\varphi(i)} \to P$ and $z_{\varphi(i)} \to z$, we see that z is a double root of P (to see this, use the fact that z is a double root of P if and only if $P(z) = P'(z) = 0$ and that if $P_{\varphi(i)} \to P$, then $P'_{\varphi(i)} \to P'$).

3.5. *Prove that the interior of* $\mathrm{D}(n, \mathbf{C})$ *in* $\mathrm{M}(n, \mathbf{C})$ *is the set of matrices having* n *distinct eigenvalues.*

First we show that if M has n distinct eigenvalues, then M is in the interior of $\mathrm{D}(n, \mathbf{C})$. We know by 2.1 that the map $M \mapsto \chi_M$ is continuous. If M has n distinct eigenvalues, then χ_M has simple roots and by 3.4 there is some open ball U around χ_M which contains only polynomials with simple roots. The set $\chi^{-1}(U)$ is an open subset of $\mathrm{M}(n, \mathbf{C})$ (since χ is continuous) containing M and only matrices with n distinct eigenvalues which are therefore in $\mathrm{D}(n, \mathbf{C})$ also. This shows that M is in the interior of $\mathrm{D}(n, \mathbf{C})$.

Now suppose that M has fewer than n distinct eigenvalues, so that there exists P such that $Q = P^{-1}MP$ is diagonal with $Q_{11} = Q_{22}$. Let Q_ϵ be the matrix Q but with $Q_{12} = \epsilon$. It is clear that $Q_\epsilon \to Q$ as $\epsilon \to 0$ and that Q_ϵ is not in $\mathrm{D}(n, \mathbf{C})$. We then have $PQ_\epsilon P^{-1} \to M$ and $PQ_\epsilon P^{-1}$ is not in $\mathrm{D}(n, \mathbf{C})$ so that M is not in the interior of $\mathrm{D}(n, \mathbf{C})$.