HOMEWORK 1 — DUE SEP 26TH

MATH 25

There are three parts to this assignment (plus a fourth "optional" one which won't be graded). Please return the three parts *separately*.

1. Sets and maps

1.1. Prove the following statements (here X, Y and Z are three sets):

(1) $(X \cap Y) \times Z = (X \times Z) \cap (Y \times Z)$;
(2) $X \setminus (Y \cup Z) = (X \setminus Y) \cap (X \setminus Z)$;
(3) $X \setminus (Y \cap Z) = (X \setminus Y) \cup (X \setminus Z)$.

1.2. Let $f : A \to B$ be a map and let Y, Z be two subsets of B. Prove the following:

(1) $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$;
(2) $f^{-1}(Y \setminus Z) = f^{-1}(Y) \setminus f^{-1}(Z)$ if $Z \subset Y$.

1.3. Let $f : A \to B$ be a map and let Y, Z be two subsets of A. Determine whether the following are true or false (give a proof or a counterexample):

(1) $f(Y \cup Z) = f(Y) \cup f(Z)$;
(2) $f(Y \cap Z) = f(Y) \cap f(Z)$.

If you're looking for a counterexample, look for something simple, for example sets with only a few elements.

1.4. If A, B are two sets, let A^B be the set of all maps $f : B \to A$. Construct a bijection between $A^{B \times C}$ and $(A^B)^C$.

2. Finite sets

2.1. Show that if A and B are finite sets, then $\mathrm{Card}(A^B) = \mathrm{Card}(A)^{\mathrm{Card}(B)}$.

Note that this holds even if one of A or B is empty. Here is why (the following is for your information, and will not be graded): given two non-empty sets A, B, you can attach to a map $f : A \to B$ its graph $G(f) \subset A \times B = \{(a, f(a))\ a \in A\}$. Show that a subset $G \subset A \times B$ is the graph of some map if and only if for any $a \in A$, $\mathrm{Card}((\{a\} \times B) \cap G) = 1$.

This definition of a "map" still makes sense if A or B is empty. How many maps are there from the empty set to itself?

2.2. Let $\binom{n}{k}$ be the number of parts of $\{1, \cdots, n\}$ which have exactly k elements. Prove that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$$

by constructing bijections between certain sets. Show also that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \cdots + \binom{k-1}{k-1} \qquad \text{and that} \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

2.3. Given $n, p \geq 1$, compute $\mathrm{Card}\{(x_1, \cdots, x_n) \in \mathbf{N}^n,\ \sum_{i=1}^n x_i = p\}$.

Hint: let $h(n,p)$ be the above cardinal. What is $h(1,p)$? Find a formula for $h(n,p)$ in terms of the $h(n-1,q)$ with $q \leq p$ and use exercise 2.2.

2.4. **The marriage lemma, after H. Weyl.** Let W and M be two finite sets. To every $m \in M$ is associated a subset $W_m \subset W$. Show that the following are equivalent:

(1) there exists an injective map $f : M \to W$ such that $f(m) \in W_m$ for every $m \in M$;
(2) for every non-empty subset $X \subset M$, $\mathrm{Card}(\cup_{m \in X} W_m) \geq \mathrm{Card}(X)$.

(recall that $\cup_{m \in X} W_m$ stands for the set of $w \in W$ such that there exists $m \in X$ with $w \in W_m$).

Conclusion: if there are men and women and if every man knows some of the women, it is possible to marry each man to a woman he knows if and only if for all n, any given n men know at least n women between each other.

3. Denumerable sets and cardinals

3.1. Finish the proof that if A and B are two denumerable sets then so is $A \times B$.

3.2. Show that $\mathbf{Q}$ is denumerable (construct an explicit bijection between $\mathbf{Q}$ and $\mathbf{Z}_{\geq 1}$).

3.3. Show that if A is any set, then $\mathrm{Card}(P(A)) = \mathrm{Card}(\{0,1\}^A)$.

3.4. Let E be an infinite set and let D be a countable subset, such that $E \setminus D$ is still infinite. Prove that $\mathrm{Card}(E) = \mathrm{Card}(E \setminus D)$.

3.5. Let A and B be two sets such that there exist two maps $f_i, f_s : A \to B$ with f_i injective and f_s surjective. Show that $\mathrm{Card}(A) = \mathrm{Card}(B)$.

Hint: you will need to use the *axiom of choice*: if I is a set and $\{E_i\}_{i\in I}$ is a collection of subsets of a set E indexed by I then there exists $f : I \to E$ such that $f(i) \in E_i$ for all $i \in I$. Sounds obvious, doesn't it?

4. Supplementary exercises

These exercises are harder and will *not* be graded. They are good practice though (especially the first one).

4.1. The purpose of this exercise is to show that $\mathrm{Card}(\mathbf{N}) \neq \mathrm{Card}(\mathbf{R})$.

(1) Show that for any real number $0 \le \alpha \le 1$, there exists a sequence $\{x_i\}_{i\ge 1}$ with $x_i \in \{0, 1\}$ such that $\alpha = \sum_{i=1}^{\infty} x_i 2^{-i}$.

Hint: if $0 \le \alpha \le 1$, then either $0 \le \alpha \le 1/2$ or $0 \le \alpha - 1/2 \le 1/2$. Suppose that you have $\{x_i\}_{1\le i\le n}$ such that $0 \le \alpha - x_1 2^{-1} - \cdots - x_n 2^{-n} \le 1/2^n$. Show that there exists $x_{n+1} \in \{0, 1\}$ such that $0 \le \alpha - x_1/2 - \cdots - x_{n+1} 2^{-(n+1)} \le 1/2^{n+1}$.

(2) Use this to define a surjective map $\{0, 1\}^{\mathbf{N}} \to [0, 1]$;

(3) Show that the map $\{0, 1\}^{\mathbf{N}} \to [0, 1]$ which sends $\{x_i\}_{i\ge 1}$ to $\sum_{i=1}^{\infty} x_i 3^{-i}$ is injective;

(4) Show that $\mathrm{Card}(P(\mathbf{N})) = \mathrm{Card}(\{0, 1\}^{\mathbf{N}}) = \mathrm{Card}([0, 1]) = \mathrm{Card}(\mathbf{R})$;

(5) Show that $\mathrm{Card}(\mathbf{N}) \neq \mathrm{Card}(\mathbf{R})$.

4.2. Construct a map $f : \mathbf{R} \to P(\mathbf{N})$ such that:

(1) for any $x \in \mathbf{R}$, $f(x)$ is infinite;

(2) for any $x \neq y \in \mathbf{R}$, $f(x) \cap f(y)$ is finite.

4.3. **Sperner's lemma.** Let A be a collection of elements of $P(\{1, \cdots, n\})$ such that for any $X, Y \in A$, one has $X \not\subset Y$.

(1) Show that $\sum_{X\in A} \mathrm{Card}(X)!(n - \mathrm{Card}(X))! \le n!$;

(2) prove Sperner's lemma: $\mathrm{Card}(A) \le \binom{n}{\lfloor n/2 \rfloor}$.

4.4. **A theorem of É. Borel.** Let $C(\mathbf{R}, \mathbf{R})$ be the set of continuous functions from $\mathbf{R}$ to $\mathbf{R}$. Show that $\mathrm{Card}(C(\mathbf{R}, \mathbf{R})) = \mathrm{Card}(\mathbf{R})$.