HOMEWORK 5 — DUE OCT 24TH

MATH 25

There are three sections: A,B and C (plus a fourth "optional" one which won't be graded). Please return the three parts *separately* to the proper CA.

A. Problems graded by Jennifer

A.1. **The AM-GM inequality.** Let $a_1, \cdots, a_n$ be positive real numbers. The arithmetic mean of the a_k's is $(a_1 + \cdots + a_n)/n$ and their geometric mean is $\sqrt[n]{a_1 \cdots a_n}$. The AM-GM inequality says that

$$\sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n}.$$

The goal of this problem is to prove the above inequality:

(1) If $a, b \geq 0$, compute $(\sqrt{a} + \sqrt{b})^2 - (\sqrt{a} - \sqrt{b})^2$ and deduce that $\sqrt{ab} \leq (a+b)/2$;

(2) Use this to prove that if $k \geq 1$ and $a_1, \cdots, a_{2^k}$ are positive real numbers, then

$$\sqrt[2^k]{a_1 \cdots a_{2^k}} \leq \frac{a_1 + \cdots + a_{2^k}}{2^k}.$$

(3) prove the AM-GM inequality. Hint: given $a_1, \cdots, a_n$ choose k such that $2^k \geq n$ and set $a_{n+1}, \cdots, a_{2^k} = \sqrt[n]{a_1 \cdots a_n}$.

A.2. Let $\{f_n\}_{n \geq 0} : \mathbf{R}_{\geq 0} \to \mathbf{R}$ be the sequence of functions defined by $f_0(x) = 0$ and

$$f_{n+1}(x) = f_n(x) + \frac{1}{2}(x - f_n^2(x)).$$

(1) Prove that for every $x \in [0, 1]$,

$$0 \leq \sqrt{x} - f_n(x) \leq \sqrt{x}\left(1 - \frac{\sqrt{x}}{2}\right)^n.$$

(2) Prove that f_n converges uniformly to $\sqrt{x}$ on $[0, 1]$.

B. Problems graded by Benjamin

B.1. **Simple convergence and uniform convergence I.** Let $I = [a, b]$ be an interval and let $\{f_n\}_{n \geq 1}$ and f be functions from I to $\mathbf{R}$. We say that the sequence $\{f_n\}_{n \geq 1}$

- converges simply to f if for every $x \in I$, we have $f_n(x) \to f(x)$.
- converges uniformly to f if $\sup_{x \in I} |f_n(x) - f(x)| \to 0$ as $n \to +\infty$.

(1) Prove that if $\{f_n\}_{n\geq 1}$ converges uniformly to f then it converges simply to f;

(2) prove that the converse is not true as follows: let $I = [0,1]$ and define

$$f_n(x) = x^n \quad \text{and} \quad f(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Show that $\{f_n\}_{n\geq 1}$ converges simply to f but that the convergence is not uniform.

(3) prove that even if f is continuous then simple convergence does not necessarily imply uniform convergence, by considering

$$f_n(x) = \begin{cases} nx & \text{if } 0 \leq x \leq 1/(2n) \\ 1 - nx & \text{if } 1/(2n) \leq x \leq 1/n \\ 0 & \text{if } 1/n \leq x \leq 1. \end{cases}$$

B.2. **Simple convergence and uniform convergence II.** Assume that $\{f_n : I \to \mathbf{R}\}_{n\geq 1}$ is a sequence of continuous functions which converges simply to $f : I \to \mathbf{R}$, where f is also continuous, and that in addition, $f_{n+1}(x) \geq f_n(x)$ for every $n \geq 1$.

(1) show that the sequence $\{\sup_{x\in I} |f_n(x) - f(x)|\}_{n\geq 1}$ is non-increasing;

(2) prove that $\{f_n\}_{n\geq 1}$ converges uniformly to f.

B.3. **Simple convergence and uniform convergence III.** Assume that $\{f_n : I \to \mathbf{R}\}_{n\geq 1}$ is a sequence of functions which converges simply to $f : I \to \mathbf{R}$, and that in addition, for every $x \in I$ and every sequence $\{x_n\}_{n\geq 1}$ which converges to x, we have $f_n(x_n) \to f(x)$

(1) prove that for every $\epsilon > 0$, there exists $\delta > 0$ and N such that $|x - y| < \delta$ implies that $|f_n(x) - f_n(y)| < \epsilon$ for all $n > N$;

(2) prove that f is continuous;

(3) prove that $\{f_n\}_{n\geq 1}$ converges uniformly to f.

C. Problems graded by Inna

C.1. **Homeomorphisms of $[0,1]$.** Let $f : [0,1] \to [0,1]$ be a continuous bijection. Prove that either $f(0) = 0$ and $f(1) = 1$ and f is strictly increasing, or $f(0) = 1$ and $f(1) = 0$ and f is strictly decreasing.

C.2. Let $f : [0,1] \to [0,1]$ be a function such that for every $x \in [0,1]$, there exists $n_x \geq 1$ with $f^{n_x}(x) = x$ (where f^n is f iterated n times).

(1) Show that f is a bijection;

(2) assume that f is continuous and that $f(0) < 1$. Find f.

C.3. **Periods of functions.** Let f, g be continuous functions from $\mathbf{R}$ to $\mathbf{R}$ and let $\alpha > 0$ be such that $f(x+1) = f(x)$ and $g(x+\alpha) = g(x)$. In (3), you can assume that 1 is the smallest period of f and that α is the smallest period of g.

(1) prove that if $\alpha \in \mathbf{Q}$ then $f + g$ is periodic;
(2) prove that if h is a continuous function which is periodic with two periods 1 and $\alpha \in \mathbf{R} \setminus \mathbf{Q}$, then h is constant;
(3) prove that $f + g$ is a periodic function if and only if $\alpha \in \mathbf{Q}$. Hint: if $f + g$ is periodic of period λ, what can you say about the function $g(x+\lambda) - g(x) = -f(x+\lambda) + f(x)$?

D. Supplementary problems

D.1. **Ascoli's theorem.** Let E be the space of all continuous functions $f : [0,1] \to \mathbf{R}$ and let $d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$. Prove that the following are equivalent for a subset K of E:

(1) K is closed, bounded, and equicontinuous (meaning: that for every $\epsilon > 0$, there exists δ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$ for all $f \in K$);
(2) K is compact.