HOMEWORK 10 — DUE DEC 5TH

MATH 25

There are two sections: A and B. Please return each part to the proper CA.

A. Problems graded by Benjamin

A.1. Find the rank of the matrix $A = (a_{i,j})_{1 \leq i,j \leq n}$ in the following cases:

(1) $a_{i,j} = i + j - 1$.
(2) $a_{i,j} = \cos(\alpha(i + j))$ where $\alpha \in \mathbf{R}$.
(3) $a_{i,j} = (\alpha + i + j)^2$ where $\alpha \in \mathbf{R}$.

A.2. Let E be a vector space over a field K, and let F be a subspace (not equal to E). Prove that $\mathrm{Vect}(E \setminus F) = E$ (i.e. that every vector in E is a linear combination of vectors of $E \setminus F$). Make sure your proof works for every field K.

A.3. Let K be an infinite field, let V be a K-vector space of dimension n and let $W_1, \cdots, W_k$ be subspaces of V of dimension $\leq n - 1$. Prove that $V \neq \cup_{i=1}^{k} W_i$.

Show that this is not always true if K is finite.

A.4. Let V be a K-vector space of dimension n and let W_1 and W_2 be two subspaces of the same dimension $r \leq n - 1$. Prove that there exists a subspace U of V such that $V = W_1 \oplus U$ and $V = W_2 \oplus U$.

A.5. Let V be a finite dimensional K-vector space and let $u, v, w \in \mathcal{L}(V, V)$ be three endomorphisms of V. Prove the following inequalities:

(1) $\mathrm{rk}(u + v) \leq \mathrm{rk}(u) + \mathrm{rk}(v)$.
(2) $\mathrm{rk}(u + v) \geq |\,\mathrm{rk}(u) - \mathrm{rk}(v)|$.
(3) $\inf(\mathrm{rk}(u), \mathrm{rk}(v)) \geq \mathrm{rk}(uv) \geq \mathrm{rk}(u) + \mathrm{rk}(v) - \dim(V)$
(4) $\mathrm{rk}(uv) + \mathrm{rk}(vw) \leq \mathrm{rk}(v) + \mathrm{rk}(uvw)$.

Use the last inequality to prove that if $k \geq 1$, and u^k denotes $u \times \cdots \times u$ (k times) then:

$$\mathrm{rk}(u^k) \leq \frac{1}{2}(\mathrm{rk}(u^{k+1}) + \mathrm{rk}(u^{k-1})).$$

B. Problems graded by Inna

B.1. Let $e = \{e_1, \cdots, e_n\}$ and $f = \{f_1, \cdots, f_n\}$ be two bases of a vector space V. Check that if A is the matrix of e in f and if B is the matrix of f in e then $AB = BA = \mathrm{Id}$.

B.2. Let V be the space of polynomials $P \in \mathbf{R}[X]$ whose degree is at most $n-1$.

(1) check that $\{1, X, X^2, \cdots, X^{n-1}\}$ is a basis of V and that the map $P(X) \mapsto P(X+1)$ is a linear map from V to itself.
(2) Write the matrix M of $P(X) \mapsto P(X+1)$ in the above basis.
(3) find M^{-1}.

B.3. Let V be the vector space of all functions from $\mathbf{R}$ to $\mathbf{R}$. Prove that the following families are free:

(1) $\{e_1 \cdots e_n\}$ where the α_i's are distinct real numbers and $e_i(x) = e^{\alpha_i x}$.
(2) $\{f_1 \cdots f_n\}$ where the α_i's are distinct real numbers and $f_i(x) = |x - \alpha_i|$.

B.4. Let U, V, W be three (finite dimensional) K-vector spaces, and let $w \in \mathcal{L}(U, W)$.

(1) If $u \in \mathcal{L}(U, V)$, prove that there exists $v \in \mathcal{L}(V, W)$ such that $w = vu$ if and only if $\ker u \subset \ker w$.
(2) If $v \in \mathcal{L}(V, W)$, prove that there exists $u \in \mathcal{L}(U, V)$ such that $w = vu$ if and only if $\operatorname{im} w \subset \operatorname{im} v$.

B.5. Let M be a $n \times n$ matrix with entries in a field K.

(1) prove that M is invertible if and only if it has rank n.
(2) prove that if we write $M = (C_1, \cdots, C_n)$ then M is invertible if and only if the dimension of the subspace of K^n generated by $C_1, \cdots, C_k$ is k for every $1 \leq k \leq n$.
(3) assume now that $K = \mathbf{Z}/p\mathbf{Z}$, the field with p elements. How many $n \times n$ matrices with entries in $\mathbf{Z}/p\mathbf{Z}$ are there?
(4) how many of them are invertible?
(5) (extra credit) let $f(n, r)$ be the number of $n \times n$ matrices with entries in $\mathbf{Z}/p\mathbf{Z}$ which have rank r. Give a formula for $f(n, r)$ and check that $f(n, 0) + f(n, 1) + \cdots + f(n, n)$ agrees with the result of (3).