SOLUTIONS FOR THE MIDTERM

MATH 25

1. An application of Taylor's expansion

In this problem, $a \in \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ is a C^∞ function. The goal of the problem is to prove that $x \mapsto (f(x) - f(a))/(x - a)$, which is defined on $\mathbf{R} \setminus \{a\}$, extends to a C^∞ function on $\mathbf{R}$.

1.1. **8pts.** *Prove that there exists a continuous map $g : \mathbf{R} \to \mathbf{R}$ such that for every $x \neq a$, we have*

$$g(x) = \frac{f(x) - f(a)}{x - a}.$$

What is $g(a)$ equal to?

Solution: The function g is continuous on $\mathbf{R} \setminus \{a\}$. If we set $g(a) = f'(a)$, then the fact that f is differentiable at $x = a$ implies that $\lim_{x \to a}(f(x) - f(a))/(x - a) = f'(a) = g(a)$ so that g is continuous at $x = a$. This way, g is continuous on $\mathbf{R}$ and in addition, $g(a) = f'(a)$.

1.2. **8pts.** *Prove that $g : \mathbf{R} \to \mathbf{R}$ is C^1 and find $g'(a)$.*

Solution: Taylor's expansion around $x = a$ tells us that:

$$f(x) = f(a) + (x - a)f'(a) + (x - a)^2 \frac{f''(a)}{2} + \frac{(x - a)^3}{2} \int_0^1 (1 - u)^2 f'''(a + u(x - a))du$$

and therefore that

$$g(x) = f'(a) + (x - a)\frac{f''(a)}{2} + \frac{(x - a)^2}{2} \int_0^1 (1 - u)^2 f'''(a + u(x - a))du$$

so that

$$\frac{g(x) - g(a)}{x - a} = \frac{f''(a)}{2} + \frac{x - a}{2} \int_0^1 (1 - u)^2 f'''(a + u(x - a))du$$

from which we deduce that $\lim_{x \to a}(g(x) - g(a))/(x - a) = f''(a)/2$. This proves that $g'(a) = f''(a)/2$.

To prove that g is C^1 we could invoke the above formulas and "differentiability under the integral" but let us do it "by hand". By applying the quotient rule, we see that g is C^1 on $\mathbf{R} \setminus \{a\}$ so in order to prove that g is C^1, we need to show that $g'(x) \to g'(a)$ as $x \to a$. The quotient rule tells us that if $x \neq a$, then

$$g'(x) = \frac{(x - a)f'(x) - (f(x) - f(a))}{(x - a)^2} = \frac{f'(x) - f'(a)}{x - a} + \frac{f'(a) - g(x)}{x - a}.$$

When $x \to a$, the first term above converges to $f''(a)$. The second term is equal to:

$$-\frac{f''(a)}{2} - \frac{x-a}{2}\int_0^1 (1-u)^2 f'''(a+u(x-a))du$$

and therefore converges to $-f''(a)/2$ as $x \to a$. This proves that $g'(x) \to g'(a)$ as $x \to a$ and therefore that $g'(x)$ is continuous at $x = a$.

1.3. **12pts.** *Assume for this question that* $f(a) = f'(a) = \cdots = f^{(n+1)}(a) = 0$, *where* $n \in \mathbf{Z}_{\geq 0}$. *Prove that* $g : \mathbf{R} \to \mathbf{R}$ *is* C^n *and that* $g(a) = g'(a) = \cdots = g^{(n)}(a) = 0$.

Solution: In this case, Taylor's expansion is just:

$$f(x) = \frac{(x-a)^{n+2}}{(n+1)!}\int_0^1 (1-u)^{n+1} f^{(n+2)}(a+u(x-a))du,$$

and therefore

$$g(x) = \frac{(x-a)^{n+1}}{(n+1)!}\int_0^1 (1-u)^{n+1} f^{(n+2)}(a+u(x-a))du.$$

Since $g(x)$ is the product of $(x-a)^{n+1}$ by the function

$$\frac{1}{(n+1)!}\int_0^1 (1-u)^{n+1} f^{(n+2)}(a+u(x-a))du$$

which is C^∞ as can be seen by differentiating under the integral sign for example, we see that g is C^n (actually C^∞) and that $g(a) = g'(a) = \cdots = g^{(n)}(a) = 0$.

1.4. **12pts.** *Return now to the case of a general* C^∞ *function* $f : \mathbf{R} \to \mathbf{R}$. *Prove that* $g : \mathbf{R} \to \mathbf{R}$ *is* C^∞ *and compute* $g^{(n)}(a)$ *for* $n \geq 0$.

Solution: We've already seen that g is C^∞; choose $n \geq 0$ and set

$$f_0(x) = f(x) - f(a) - (x-a)f'(a) - \cdots - \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(a)$$

so that $f_0(a) = f_0'(a) = \cdots = f_0^{(n+1)}(a) = 0$. We have

$$g_0(x) = g(x) - f'(a) - \cdots - \frac{(x-a)^n}{(n+1)!}f^{(n+1)}(a)$$

and by the previous question, we know that $g_0(a) = g_0'(a) = \cdots = g_0^{(n)}(a) = 0$. We see that $g_0^{(n)}(a) = g^{(n)}(a) - \frac{n!}{(n+1)!}f^{(n+1)}(a)$ so that $g^{(n)}(a) = f^{(n+1)}(a)/(n+1)$.

2. Derivations of $C^\infty(\mathbf{R}, \mathbf{R})$

In this problem, we will use the results from problem 1. Let $C^\infty(\mathbf{R}, \mathbf{R})$ denote the set of all maps $f : \mathbf{R} \to \mathbf{R}$ which are C^∞. The goal of this problem is to find all maps $T : C^\infty(\mathbf{R}, \mathbf{R}) \to C^\infty(\mathbf{R}, \mathbf{R})$ satisfying the following three properties:

(1) $T(f+g) = T(f) + T(g)$;
(2) $T(\lambda f) = \lambda T(f)$ if $\lambda \in \mathbf{R}$;
(3) $T(fg) = fT(g) + gT(f)$.

Observe that $T(\varphi) = \varphi'$ is one such map.

2.1. **8pts.** *Let 1 denote the constant function $x \mapsto 1$. What is $T(1)$?*

Solution: We have $T(1) = T(1 \cdot 1) = 1T(1) + 1T(1) = 2T(1)$ so $T(1) = 0$.

2.2. **8pts.** *For $a \in \mathbf{R}$, let $\theta_a : \mathbf{R} \to \mathbf{R}$ be the map $x \mapsto (x-a)^2$ and let $\varepsilon_a = T(\theta_a)$. What is $\varepsilon_a(a)$?*

Solution: We have $T((x-a)^2) = 2(x-a)T(x-a)$ so that $\varepsilon_a(a) = 0$.

2.3. **12pts.** *Let $\varphi \in C^\infty(\mathbf{R}, \mathbf{R})$ be a function such that $\varphi'(a) = 0$. Prove that $T(\varphi)(a) = 0$.*

By the preceding problem, we know that the function $\psi(x) = (\varphi(x) - \varphi(a))/(x-a)$ is C^∞ and that $\psi(a) = 0$. So we apply the preceding problem once again and deduce that $\chi(x) = (\varphi(x) - \varphi(a))/(x-a)^2$ is C^∞ so that we can write $\varphi(x) = \varphi(a) + \theta_a(x)\chi(x)$. Therefore we have $T(\varphi) = T(\theta_a)\chi + \theta_a T(\chi)$ and since $T(\theta_a)(a) = \theta_a(a) = 0$, we have $T(\varphi)(a) = 0$.

2.4. **12pts.** *Prove that there exists $f \in C^\infty(\mathbf{R}, \mathbf{R})$ such that for all $\varphi \in C^\infty(\mathbf{R}, \mathbf{R})$, we have $T(\varphi) = f\varphi'$.*

Solution: observe that $T(x-a)$ does not depend on a since $T(a) = 0$. Set $f = T(x-a)$. The function $\varphi(x) - \varphi'(a)(x-a)$ has its first derivative $= 0$ when $x = a$ so that

$$T[\varphi(x) - \varphi'(a)(x-a)](a) = 0$$

by the preceding question. This implies that $T(\varphi)(a) = \varphi'(a)f(a)$. This is true for all $a \in \mathbf{R}$, so we have $T(\varphi) = f\varphi'$.

3. A characterization of ruled functions

Let $I = [a, b]$ be a closed interval, and let $f : I \to \mathbf{R}$ be a function.

The goal of this problem is to prove that a function $f : I \to \mathbf{R}$ is ruled if and only if at every $x \in I$, f has a left limit and a right limit (at $x = a$ we only ask for a right limit and at $x = b$ we only ask for a left limit).

Let $L(I)$ be the set of functions $f : I \to \mathbf{R}$ such that at every $x \in I$, f has a left limit and a right limit (at $x = a$ we only ask for a right limit and at $x = b$ we only ask for a left limit). Let $S(I)$ be the set of step functions $f : I \to \mathbf{R}$. If $f : I \to \mathbf{R}$ is a bounded function, let $|f|_I$ denote $\sup_{x \in I} |f(x)|$.

3.1. **12pts.** *Prove that if $f \in L(I)$, then f is bounded.*

Solution: if this was not the case, there would exist a sequence $\{x_n\}_{n \geq 1}$ of elements of I such that $f(x_n) \to +\infty$. Since I is compact, we can extract from x_n a subsequence $x_{\varphi(n)}$ converging to some $x \in I$. By extracting again, we can assume that either $x_{\varphi \circ \psi(n)} > x$ or $x_{\varphi \circ \psi(n)} < x$ for all n. Let us assume that the first case holds. Since $x_{\varphi \circ \psi(n)} > x$ and $x_{\varphi \circ \psi(n)} \to x$, we have $f(x_{\varphi \circ \psi(n)}) \to y^+$ where y^+ is the right limit of f at x. This contradicts the fact that the $f(x_{\varphi \circ \psi(n)})$ are unbounded. Therefore, if $f \in L(I)$, then f is bounded.

3.2. **12pts.** *Prove that if one defines $d(f, g) = |f - g|_I$ for f, $g \in L(I)$, then $(L(I), d)$ is a complete metric space.*

Solution: It is easy to check that d satisfies the three requirements to be a distance on $L(I)$, so we will now prove that $(L(I), d)$ is complete. We have to prove that if $\{f_n\}_{n \geq 1}$ is a sequence of elements of $L(I)$ which converges uniformly to a function f, then f has a right limit and a left limit at each $x \in I$. Let us do the case of a left limit, the case of a right limit being analogous. Choose $x \in I$ and a sequence $\{x_n\}_{n\geq 1}$ such that $x_n < x$ and $x_n \to x$. Choose $\varepsilon > 0$. There exists $m \geq 0$ such that $|f - f_m|_I < \varepsilon$. Since $f_m(x_n) \to y^-$ as $n \to \infty$, there exists $N \geq 0$ such that if $k, \ell \geq N$ then $|f_m(x_k) - f_m(x_\ell)| < \varepsilon$. Now we have

$$|f(x_k) - f(x_\ell)| < |f(x_k) - f_m(x_k)| + |f_m(x_k) - f_m(x_\ell)| + |f_m(x_\ell) - f(x_\ell)|,$$

so that $|f(x_k) - f(x_\ell)| < 3\varepsilon$ if $k, \ell \geq N$. This proves that for every sequence $\{x_n\}_{n\geq 1}$ such that $x_n < x$ and $x_n \to x$, $f(x_n)$ has a limit. By "intertwining" sequences, it is clear that this limit is unique and therefore that f has a left limit at x. This proves that $f \in L(I)$ and therefore that $(L(I), d)$ is a complete metric space.

3.3. **12pts.** *Explain why $S(I) \subset L(I)$. Prove that $(S(I), d)$ is dense in $(L(I), d)$. Hint: prove that if $f \in L(I)$ and $\varepsilon > 0$, then the set of $x \in I$ such that there exists $\varphi \in S(I)$ satisfying $\sup_{y\in[a,x]} |f(y) - \varphi(y)| \leq \varepsilon$ is both open and closed.*

Solution: $S(I) \subset L(I)$ because a step function has a left limit and a right limit at every point (this is clear from the definition). Let us now prove that $(S(I), d)$ is dense in $(L(I), d)$. We will first show that this follows from the hint. Choose $\varepsilon > 0$; if the set of $x \in I$ such that there exists $\varphi \in S(I)$ satisfying $\sup_{y\in[a,x]} |f(y) - \varphi(y)| \leq \varepsilon$ is both open and closed, then it is a connected component of I and it has to be empty or I itself since I is connected. Since this set contains a at least, we see that it is I itself. By the definition of this set, there exists $\varphi \in S(I)$ satisfying $|f(y) - \varphi(y)|_I \leq \varepsilon$. This shows that f is a uniform limit of elements of $S(I)$.

Now, we will prove the hint. Let $A(f, \varepsilon)$ be the set of $x \in I$ such that there exists $\varphi \in S(I)$ satisfying $\sup_{y\in[a,x]} |f(y) - \varphi(y)| \leq \varepsilon$. If $x \in A(f, \varepsilon)$, then obviously any $x' \leq x$ is in $A(f, \varepsilon)$ also; this implies that $A(f, \varepsilon) = [a, x)$ or that $A(f, \varepsilon) = [a, x]$ for some x. We will first show that $A(f, \varepsilon)$ is open, so suppose that $A(f, \varepsilon) = [a, x]$. If $x = b$, we're done. Otherwise, f has a right limit y^+ at x. We claim that this implies that there exists $\delta > 0$ such that if $x < z < x + \delta$ then $|f(z) - y^+| < \varepsilon/2$. If this was not the case, we could find a sequence z_n violating the claim and converging to x, which would contradict the fact that f has a right limit at x. The claim is therefore true, and if we have φ such that $\sup_{y\in[a,x]} |f(y) - \varphi(y)| \leq \varepsilon$, then we can extend φ to $[a, x + \delta)$ by setting $\varphi = y^+$ on $(x, x + \delta)$ and $\varphi(x) = f(x)$. We now prove that $A(f, \varepsilon)$ is closed. Suppose that there exists x and some sequence $x_n \to x$ and step functions φ_n on $[a, x_n]$ such that $\sup_{y\in[a,x_n]} |f(y) - \varphi_n(y)| \leq \varepsilon$. Since f has a left limit y^- at x, an argument similar to the one above shows that there exists $\delta > 0$ such that if $x - \delta < z < x$ then $|f(z) - y^-| < \varepsilon/2$ and if we choose n such that $x_n > x - \delta$, then the step function φ defined by $\varphi = \varphi_n$ on $[a, x_n]$ and $\varphi = y^-$ on (x_n, x) and $\varphi = f(x)$ at x is within ε of f and therefore, $x \in A(f, \varepsilon)$. This proves the hint.

3.4. **4pts.** *Conclude: prove that $L(I)$ is equal to the set of ruled functions on I.*

Solution: a function is ruled if and only if it is a uniform limit of elements of $S(I)$ and so if and only if it is in $L(I)$ so that $L(I)$ coincides with the set of all ruled functions.