# MIDTERM

MATH 25

INSTRUCTIONS: READ CAREFULLY

This midterm is due MONDAY NOVEMBER 3RD AT NOON. You should return it in class Monday, the noon deadline being there to avoid catastrophes (alarm clock not working, too much candy after Halloween, etc). If you return the midterm after class, put it in my mailbox.

- You **are** allowed to use: your notes from class, anything that's on the course's website or on the CA's website.
- You **are not** allowed to use: (text)books, photocopies, anything that's on the web, a calculator or a computer. You are not allowed to discuss the midterm with your classmates or anybody else (except your CA's and me).

I have read the above and agree to it:

Name:________________________________________ Signature:________________________________________

Attach this to your solutions. I will only accept paper (not emailed) solutions.

| § | 1.1 | 1.2 | 1.3 | 1.4 | 2.1 | 2.2 | 2.3 | 2.4 | 3.1 | 3.2 | 3.3 | 3.4 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Points | | | | | | | | | | | | | |
| Out of | 8 | 8 | 12 | 12 | 8 | 8 | 12 | 12 | 12 | 12 | 12 | 4 | 120 |

## 1. AN APPLICATION OF TAYLOR'S EXPANSION

In this problem, $a \in \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ is a $C^\infty$ function. The goal of the problem is to prove that $x \mapsto (f(x) - f(a))/(x - a)$, which is defined on $\mathbf{R} \setminus \{a\}$, extends to a $C^\infty$ function on $\mathbf{R}$.

1.1. **8pts.** Prove that there exists a continuous map $g : \mathbf{R} \to \mathbf{R}$ such that for every $x \neq a$, we have

$$g(x) = \frac{f(x) - f(a)}{x - a}.$$

What is $g(a)$ equal to?

1.2. **8pts.** Prove that $g : \mathbf{R} \to \mathbf{R}$ is $C^1$ and find $g'(a)$.

1.3. **12pts.** Assume for this question that $f(a) = f'(a) = \cdots = f^{(n+1)}(a) = 0$, where $n \in \mathbf{Z}_{\geq 0}$. Prove that $g : \mathbf{R} \to \mathbf{R}$ is $C^n$ and that $g(a) = g'(a) = \cdots = g^{(n)}(a) = 0$.

1.4. **12pts.** Return now to the case of a general $C^\infty$ function $f : \mathbf{R} \to \mathbf{R}$. Prove that $g : \mathbf{R} \to \mathbf{R}$ is $C^\infty$ and compute $g^{(n)}(a)$ for $n \geq 0$.

## 2. Derivations of $C^\infty(\mathbf{R}, \mathbf{R})$

In this problem, we will use the results from problem 1. Let $C^\infty(\mathbf{R}, \mathbf{R})$ denote the set of all maps $f : \mathbf{R} \to \mathbf{R}$ which are $C^\infty$. The goal of this problem is to find all maps $T : C^\infty(\mathbf{R}, \mathbf{R}) \to C^\infty(\mathbf{R}, \mathbf{R})$ satisfying the following three properties:

(1) $T(f + g) = T(f) + T(g)$;
(2) $T(\lambda f) = \lambda T(f)$ if $\lambda \in \mathbf{R}$;
(3) $T(fg) = fT(g) + gT(f)$.

Observe that $T(\varphi) = \varphi'$ is one such map.

2.1. **8pts.** Let 1 denote the constant function $x \mapsto 1$. What is $T(1)$?

2.2. **8pts.** For $a \in \mathbf{R}$, let $\theta_a : \mathbf{R} \to \mathbf{R}$ be the map $x \mapsto (x - a)^2$ and let $\varepsilon_a = T(\theta_a)$. What is $\varepsilon_a(a)$?

2.3. **12pts.** Let $\varphi \in C^\infty(\mathbf{R}, \mathbf{R})$ be a function such that $\varphi'(a) = 0$. Prove that $T(\varphi)(a) = 0$.

2.4. **12pts.** Prove that there exists $f \in C^\infty(\mathbf{R}, \mathbf{R})$ such that for all $\varphi \in C^\infty(\mathbf{R}, \mathbf{R})$, we have $T(\varphi) = f\varphi'$.

## 3. A characterization of ruled functions

Let $I = [a, b]$ be a closed interval, and let $f : I \to \mathbf{R}$ be a function. We say that $f$ has a left limit at $x \in I$ if there exists $y^- \in \mathbf{R}$ such that for every sequence $\{x_n\}_{n \geq 1}$ satisfying $x_n < x$ and $x_n \to x$, we have $f(x_n) \to y^-$. The definition of right limits $y^+$ is analogous.

The goal of this problem is to prove that a function $f : I \to \mathbf{R}$ is ruled if and only if at every $x \in I$, $f$ has a left limit and a right limit (at $x = a$ we only ask for a right limit and at $x = b$ we only ask for a left limit).

Let $L(I)$ be the set of functions $f : I \to \mathbf{R}$ such that at every $x \in I$, $f$ has a left limit and a right limit (at $x = a$ we only ask for a right limit and at $x = b$ we only ask for a left limit). Let $S(I)$ be the set of step functions $f : I \to \mathbf{R}$. If $f : I \to \mathbf{R}$ is a bounded function, let $|f|_I$ denote $\sup_{x \in I} |f(x)|$.

3.1. **12pts.** Prove that if $f \in L(I)$, then $f$ is bounded.

3.2. **12pts.** Prove that if one defines $d(f, g) = |f - g|_I$ for $f$, $g \in L(I)$, then $(L(I), d)$ is a complete metric space.

3.3. **12pts.** Explain why $S(I) \subset L(I)$. Prove that $(S(I), d)$ is dense in $(L(I), d)$. Hint: prove that if $f \in L(I)$ and $\varepsilon > 0$, then the set of $x \in I$ such that there exists $\varphi \in S(I)$ satisfying $\sup_{y \in [a,x]} |f(y) - \varphi(y)| \leq \varepsilon$ is both open and closed.

3.4. **4pts.** Conclude: prove that $L(I)$ is equal to the set of ruled functions on $I$.