2.1. *Show that if A and B are finite sets,* $\mathrm{Card}(A^B) = \mathrm{Card}(A)^{\mathrm{Card}(B)}$.

Solution. (Due to Eric Suh.) Let $\mathrm{Card}(A) = m$, $\mathrm{Card}(B) = n$; denote the elements of A and B by $A = \{a_1, \ldots, a_m\}$, and $B = \{b_1, \ldots, b_n\}$. Obvious as it is, we can "rigorously" show that A^n has cardinality m^n by constructing a bijection $f : A^n \to \overline{J}_{m^n}$, where $\overline{J}_{m^n} = \{0, \ldots, m^n - 1\}$, by sending $(a_{i_1}, \ldots, a_{i_n})$ to a number whose base m expansion has digits $i_1 \cdots i_n$; every number between 0 and $m^n - 1$ has a unique base m expansion with n digits, so f is bijective. We can then map $g : A^B \to A^n$ by sending $\phi : B \to A$ to $(x_1, \ldots, x_n)$, where $x_i = \phi(b_i)$. g is surjective, as every n-tuple x_i defines a function $\phi(i) = x_i$; and if for two $\phi, \psi \in A^B$, $(x_i) = g(\phi) = g(\psi) = (y_i)$, then $\phi(b_i) = x_i = y_i = \psi(b_i)$ for all $i \in J_n$ and $\psi = \phi$. thus, $\mathrm{Card}(A^B) = \mathrm{Card}(\overline{J}_{m^n}) = m^n$. □

2.2. (i) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

Solution. For any $S \in P_k(J_n)$, if $n \notin S$ we may consider S as an element of $P_k(J_{n-1})$, while if $n \in S$, $S \setminus \{n\} \in P_{k-1}(J_{n-1})$. Define a function $f : P_k(J_n) \to P_k(J_{n-1}) \sqcup P_{k-1}(J_{n-1})$ by

$$S \in P_k(J_n), \quad f(S) = \begin{cases} S & n \notin S \\ S \setminus \{n\} & n \in S \end{cases}$$

For any $S \in P_k(J_{n-1}) \sqcup P_{k-1}(J_{n-1})$, either $S \in P_k(J_{n-1})$ or $S \in P_{k-1}(J_{n-1})$; in the former case, $f(S) = S$, while in the latter, $f(S \setminus \{n\}) = S$, and thus f is surjective. For $S_1, S_2 \in P_k(J_n)$, suppose $f(S_1) = f(S_2)$. If $f(S_1) = f(S_2)$ has cardinality k, then n is in neither S_1 nor S_2, and $S_1 = S_2$, while if it has cardinality $k - 1$, then n is in both S_1 and S_2, so $S_1 = f(S_1) \cap \{n\} = f(S_2) \cap \{n\} = S_2$. f is therefore bijective, and the claim follows. □

(ii) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \cdots + \binom{k}{k-1} + \binom{k-1}{k-1}$

Solution. Certainly $\binom{n}{0} = 1 = \binom{n}{n}$. Assuming the formula holds for $\binom{m}{l}$, $m + 1 < n + k$, we have by the first part of the problem

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n-1}{k-1} + \cdots + \binom{k-1}{k-1}$$

and by induction on n, we prove the result. □

(iii) $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Solution. Once again, $\binom{n}{0} = 1 = \binom{n}{n}$. If we assume the result is true for $n - 1$ and all k,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!}$$
$$= \frac{n!}{k!(n-k)!}\left(\frac{k}{n} + \frac{n-k}{n}\right) = \frac{n!}{k!(n-k)!}$$

□

2.3. *Given $n, p \geq 1$ compute* $\mathrm{Card}\{(x_1, \ldots, x_n) \in \mathbb{N}^n \mid \sum_{i=1}^n x_i = p\}$.

Solution. We will prove that $h(n,p) = \binom{p+n-1}{n-1}$ by induction on n. For any p, the number of ways $h(1,p)$ to write p as a sum of one number is 1, $p = p$. Assuming the result for $h(n-1,p)$, note that an n-tuple $(x_1,\ldots,x_n)$ satisfies $\sum_{i=1}^{n} x_i = p$ only if $\sum_{i=1}^{n-1} x_i$ is a number q between 0 and p, and conversely for any $n-1$-tuple $(x_1,\ldots,x_{n-1})$ summing to a number $p-q$, $0 \le q \le p$, we get an n-tuple $(x_1,\ldots,x_{n-1},q)$ whose sum is p. There are $h(n-1,q)$ such n-tuples, so by the induction hypothesis

$$h(n,p) = \sum_{q=0}^{p} h(n-1,q) = \sum_{q=0}^{p} \binom{n+q-2}{n-2}$$

$$= \binom{n+p-2}{n-2} + \binom{n+p-3}{n-2} + \cdots + \binom{n-1}{n-2} + \binom{n-2}{n-2} = \binom{n+p-1}{n-1}$$

where the last line follows from 2.2. □

If you did this problem assuming $0 \notin \mathbb{N}$, then by the same reasoning you would arrive at $h(n,p) = \binom{p-1}{n-1}$.

2.4. **The marriage lemma.** *Let W and M be finite sets. To every $m \in M$ is associated a subset $W_m \subset W$. Show that the following are equivalent:*

(1) There is an injection $f : M \to W$ such that $f(m) \in W_m$ for every $m \in M$;

(2) For every nonempty $X \subset M$, $\mathrm{Card}(\cup_{m\in X} W_m) \ge \mathrm{Card}(X)$.

Solution. (Due to Lisa Leung, Steve Howard, and Jamie Rubin) (1) $\Rightarrow$ (2). If $f : M \to W$ is an injection, then for any $X \subset M$, $\mathrm{Card}(X) = \mathrm{Card}(f(X))$. Certainly $f(X) \subset \cup_{m\in X} W_m$, so $\mathrm{Card}(\cup_{m\in X} W_m) \ge \mathrm{Card}(X)$.
(2) $\Rightarrow$ (1). We prove the result be induction on $\mathrm{Card}(M)$. For $\mathrm{Card}(M) = 1$ (say $M = \{m\}$), take any $w \in W_m$, and $f(m) = w$ defines an injective function.
Let $\mathrm{Card}(M) = n$, and assume (2) $\Rightarrow$ (1) for all M with $\mathrm{Card}(M) < n$. To prove the induction step, we need the following lemma:

Lemma. *Let $B \subset M$ be such that $\mathrm{Card}\,(\cup_{m\in B} W_m) = \mathrm{Card}(B)$. Then $\overline{M} = M \setminus B$ and $\overline{W}_m = W_m \setminus (\cup_{m\in B} W_m)$ satisfy (2), as do B and W_m.*

Proof. Suppose $\mathrm{Card}(B) = k$. For any nonempty $X \subset \overline{M}$, we clearly have

$$\bigcup_{m\in X} \overline{W}_m = \left(\bigcup_{m\in X\cup B} W_m\right) \setminus \left(\bigcup_{m\in B} W_m\right)$$

but $\mathrm{Card}(\cup_{m\in X\cup B} W_m) \ge \mathrm{Card}(X \cup B)$, so that

$$\mathrm{Card}\left(\bigcup_{m\in X} \overline{W}_m\right) \ge \mathrm{Card}(X\cup B) - \mathrm{Card}(B) = \mathrm{Card}(X)$$

For any $X \subset B$, the fact that

$$\mathrm{Card}\left(\bigcup_{m\in X} W_m\right) \ge \mathrm{Card}(X)$$

follows because M satisfies (2). □

Thus, if there is a proper (nonempty) subset $B \subset M$ satisfying the hypothesis of the lemma, then $\text{Card}(\overline{M})$ and $\text{Card}(B)$ are both less than n, so by induction there exist injections $\overline{M} \to \cup_{m \in \overline{M}} \overline{W}_m$ and $B \to \cup_{m \in B} W_m$. The images of the two functions are disjoint by definition, so we may juxtapose these two maps to get an injection $M \to W$.
If there is no such subset, then for any proper (again, nonempty) subset X,

$$\text{Card}\left(\bigcup_{m \in X} W_m\right) \geq \text{Card}(X) + 1$$

Note that if (2) holds, then for any $m_0 \in M$, $\text{Card}(W_{m_0}) \geq \text{Card}(\{m_0\}) = 1$, so we may pick an arbitrary $w_0 \in W_m$. Let $\overline{M} = M \setminus \{m_0\}$, $\overline{W}_m = W_m \setminus \{w_0\}$. For any $X \subset \overline{M}$, we then have

$$\text{Card}\left(\bigcup_{m \in X} \overline{W}_m\right) \geq \text{Card}\left(\bigcup_{m \in X} W_m\right) - 1 \geq \text{Card}X$$

by the above. $\overline{M}$ satisfies (2), so there is an injection $f : \overline{M} \to W \setminus \{w_0\}$, and by taking $f(m_0) = w_0$, we get an injection $f : M \to W$. □