A.1 [1] Give an example of a metric space (E, d) and infinitely many open sets $\{U_i \subset E\}_{i \in I}$ such that $\bigcap_{i \in I} U_i$ is not open. You could consider suitable intervals in $\mathbb{R}$ for example.

Proof. Take E to be $\mathbb{R}$, under the standard metric $d(x, y) = |x - y|$ for $x, y \in E$. Then, let U_i be the open interval $(1 - \frac{1}{n}, 1 + \frac{1}{n})$. Then, as $i \to \infty$, $(1 - \frac{1}{n}, 1 + \frac{1}{n}) \to [1, 1]$, which is clearly not open in $\mathbb{R}$. $\square$

A.2 Let X be a subset of a metric space (E, d). Show that:

$$\mathring{\overline{X}} = \mathring{\overline{\mathring{\overline{X}}}} \quad \text{and that} \quad \overline{\mathring{X}} = \overline{\mathring{\overline{\mathring{X}}}}.$$

Give an example of a subset X of $\mathbb{R}$ such that the following sets:

$$X, \overline{X}, \mathring{X}, \mathring{\overline{X}}, \overline{\mathring{X}}, \overline{\mathring{\overline{X}}}, \mathring{\overline{\mathring{X}}},$$

are pairwise distinct.

Proof of Part 1. Let $\alpha \in \mathring{\overline{X}}$. Then, for all $F \subset E$, where $F \supset \mathring{\overline{X}}$, $\alpha \in F$. Thus, $\alpha \in \overline{\mathring{\overline{X}}}$. Since α was arbitrary, $\mathring{\overline{X}} \subset \overline{\mathring{\overline{X}}}$. Thus, since $\mathring{\overline{X}}$ is open and a subset of $\overline{\mathring{\overline{X}}}$, it is also a subset of $\mathring{\overline{\mathring{\overline{X}}}}$ (which, by definition, is the union of all open subsets of $\overline{\mathring{\overline{X}}}$).
The interior of $\overline{X}$ is the same as the complement of the closure of the complement of $\overline{X}$ (in fact, this is true for all subsets in general). Thus, $\mathring{\overline{X}} = \overline{\overline{X}^C}^C$ and $\mathring{\overline{\mathring{\overline{X}}}} = \overline{\overline{\overline{\overline{X}^C}^C}}^C$. Now, $\overline{X}^C \subset \overline{\overline{X}^C}$, by definition, so $\overline{X} \supset \overline{\overline{X}^C}^C$, and thus $\overline{X} \supset \overline{\overline{\overline{X}^C}^C}$. Thus,

$$\implies \overline{X}^C \subset \overline{\overline{\overline{X}^C}^C}^C$$

$$\implies \overline{\overline{X}^C} \subset \overline{\overline{\overline{\overline{X}^C}^C}^C}$$

$$\implies \overline{\overline{X}^C}^C \supset \overline{\overline{\overline{\overline{X}^C}^C}^C}^C$$

Thus, $\mathring{\overline{X}} \supset \mathring{\overline{\mathring{\overline{X}}}}$. $\therefore \mathring{\overline{X}} = \mathring{\overline{\mathring{\overline{X}}}}$ $\square$

Part Two. We have

$$\overline{\overline{X}^C}^C = \overline{\overline{\overline{\overline{X}^C}^C}^C}^C$$

[1] Many thanks to Eric Suh for making his LaTeXfile available to my disposal.

for any $X \subset (E, d)$, and we want to show that

$$\overline{\mathring{X}} = \overline{\overline{X^C}^C} = \overline{\overline{\overline{\overline{X^C}^C}^C}^C} = \overline{\mathring{\overline{\mathring{X}}}}$$

We know that

$$\overline{\overline{(X^C)}^C}^C = \overline{\overline{\overline{\overline{(X^C)}^C}^C}^C}^C$$

since $X^C \subset (E, d)$. Therefore,

$$\overline{\overline{X^C}^C} = \overline{\overline{\overline{\overline{X^C}^C}^C}^C}$$

And thus

$$\overline{\mathring{X}} = \overline{\mathring{\overline{\mathring{X}}}}$$

□

Example Set. Let $X = (1, 3]\backslash\{2\} \cup ([4, 5] \cap \mathbb{Q}) \cup \{0\} \subset \mathbb{R}$.
$\overline{X} = [1, 3] \cup [4, 5] \cup \{0\}$
$\mathring{X} = (1, 2) \cup (2, 3)$
$\mathring{\overline{X}} = (1, 3) \cup (4, 5)$
$\overline{\mathring{X}} = [1, 3]$
$\overline{\mathring{\overline{X}}} = [1, 3] \cup [4, 5]$
$\mathring{\overline{\mathring{X}}} = (1, 3)$ □

A.3 [Induced Topology] Let (E, d) be a metric space, and let X be a subset of E so that (X, d) is itself a topological space with the induced distance. Show that $U \subset X$ is open in X if and only if there exists an open set V of E such that $U = X \cap V$.
Be careful: if $U \subset X$ is open in X, it is not necessarily open in E.

Proof. Let $U \subset X$ be open in X. For every point $x \in U$, there exists an open ball of radius r_x: $B_x(x, r_x) \subset U$. Now, consider the same open balls, but consider them as open balls in E instead of X. Then, let V be the union of all such open balls. V is open in E, and $U = X \cap V$.

Now going in the other direction, let V be an open set in E such that $U = X \cap V$. For all $x \in U$, there exists $B_E(x, r) \subset V$. Construct $B_X(x, r) = B_E(x, r) \cap X$. This is an open ball in X, and since this exists for all $x \in U$, U is open in X. □

A.4 [Product Topology] Let (E_1, d_1) and (E_2, d_2) be two metric spaces. Define a distance d on $E_1 \times E_2$ by

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2).$$

1. Show that $U \subset E_1 \times E_2$ is open if and only if for each point $(u_1, u_2) \in U$, there exists U_1 and U_2 open in E_1 and E_2 with $(u_1, u_2) \in U_1 \times U_2$ and $U_1 \times U_2 \subset U$.

2. If X is another metric space, show that $(f_1, f_2) : X \to E_1 \times E_2$ is continuous if and only if f_1 and f_2 both are.

Proof of Part 1. Let $U \subset E_1 \times E_2$ be open, and let $(u_1, u_2) \in U$. Since U is open, there exists $r > 0$ such that $B\left((u_1, u_2), r\right) \subset U$. Thus, for any (v_1, v_2) in the open ball around (u_1, u_2),

$$d((u_1, u_2), (v_1, v_2)) = d_1(u_1, v_1) + d_2(u_2, v_2) < r.$$

Clearly, we see that this implies $d_1(u_1, v_1) < r$ and $d_2(u_2, v_2) < r$. Now let U_1 be the open ball in E_1 around u_1 with radius $r/2$ and U_2 be the open ball in E_2 around u_2 with radius $r/2$. By definition, U_1 and U_2 are open. $U_1 \times U_2 \subset U$, since anything within the cartesian product must lie within the open ball in U around (u_1, u_2).

Now, assume that for each point $(u_1, u_2) \in U$ there exists U_1 and U_2 open in E_1 and E_2 with $(u_1, u_2) \in U_1 \times U_2$ and $U_1 \times U_2 \subset U$. Then there is an open ball in U_1 around u_1 with radius r_1, and an open ball in U_2 around u_2 with radius r_2. Let $r = \min(r_1, r_2)$. Then, for any element $(v_1, v_2) \in U$ such that $d\left((u_1, u_2), (v_1, v_2)\right) < r$ is within $U_1 \times U_2$. Thus, the open ball resides completely in U, and since such an open ball exists for every point (u_1, u_2) in U, U is open. □

Proof of Part 2. For all $x \in X$, and for all $\varepsilon > 0$, there is a $\delta > 0$ such that for all $x' \in X$, $d(x, x') < \delta \implies d_1(f_1(x), f_1(x')) + d_2(f_2(x), f_2(x')) < \varepsilon$. Thus, for all $x' \in X$,

$$d(x, x') < \delta \implies d_1(f_1(x), f_1(x)') < \varepsilon$$

and

$$d(x, x') < \delta \implies d_2(f_2(x), f_2(x)') < \varepsilon$$

For all $x \in X$ and for all $\varepsilon > 0$, we must find a $\delta > 0$ such that for all $x' \in X$, $d(x, x') < \delta \implies d_1(f_1(x), f_1(x)') + d_2(f_2(x), f_2(x)') < \varepsilon$. Since f_1 and f_2 are both continuous, for some $\varepsilon/2$, there is a $\delta_1 > 0$ such that $d_1(f_1(x), f_1(x)') < \varepsilon/2$ if $d(x, x') < \delta_1$, and for some $\varepsilon/2$, there is a $\delta_2 > 0$ such that $d_2(f_2(x), f_2(x)') < \varepsilon/2$ if $d(x, x') < \delta_2$. Now, let $\delta = \min(\delta_1, \delta_2)$. Then,

$$d(x, x') < \delta \implies d_1(f_1(x), f_1(x')) + d_2(f_2(x), f_2(x')) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

□