Problem Set #2 Part C
Official Solutions

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1.

(10) (i) We will first show that if f is continuous and $\{x_n\}$ converges to x then $\{f(x_n)\} \to \infty$ as $n \to \infty$. Since f is continuous at x we know that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$. Note that since $\{x_n\} \to x$ then for δ there exists an integer $N > 0$ such that if $n \geq N$ then $d(x_n, x) < \delta$. From this we see that if $n \geq N$, $d(f(x_n), f(x)) < \epsilon$, so $\{f(x_n)\} \to f(x)$.

Now we need to show that if every sequence such that $\{x_n\} \to x$ also has $\{f(x_n)\} \to f(x)$ then f is continuous.

Solution 1: Suppose that there is some x such that f is not continuous at x. That means that there exists $\epsilon > 0$ such that for all $\delta > 0$ there exists x' such that $d(x, x') < \delta$ but $d(f(x), f(x')) \geq \epsilon$. Define a sequence $\{x_n\}$ in the following way: setting $\delta = 1/n$ let x_n be such that $d(x, x_n) < \delta$ but $d(f(x), f(x_n)) \geq \epsilon$. Then this sequence converges to x. However, it does not converge to $f(x)$, since $f(x_n)$ is at least ϵ from $f(x)$ for all n. Contradiction. Thus f must be continuous.

Solution 2: We know that f is continuous if and only if the preimage of any closed set is closed. Let $F \subset Y$ be closed. We will show that $f^{-1}(F)$ is closed. Consider any sequence $\{x_n\}$ in $f^{-1}(F)$. If we can show that $x \in f^{-1}(F)$ we will be done, since a closed set is exactly one that contains all of its limit points. Consider $\{f(x_n)\}$ in F. F is closed, so we know that $f(x)$ is in F, so $x \in f^{-1}(F)$. Thus $f^{-1}(F)$ is closed, so f is continuous.

(10) (ii) This function is continuous at every irrational and discontinuous at every rational.

First we will show that it is discontinuous at every rational. Let $\{x_n\}$ be a sequence of irrationals converging to any rational p/q. Then $\{f(x_n)\}$ is identically 0. Thus it converges to 0. However, $f(p/q) = 1/q > 0$, so f is discontinuous at every rational.

Now consider any irrational $x \in [0, 1]$. Consider any sequence $\{x_n\}$ converging to x. If it has only a finite number of rationals then we can simply consider the subsequence that consists only of irrationals; this will converge to the limit of the sequence. Then $\{f(x_n)\}$ will clearly converge to 0, which is $f(x)$. Now suppose that the sequence has infinitely many rationals. We can split this sequence up into two: one with all irrationals, one with all rationals. Let $\{y_n\}$ be the subsequence of rationals, and $\{z_n\}$ be the subsequence of rationals. If $\{y_n\}$ is finite we can disregard it. If it does not, we have already shown that $\{f(y_n)\}$ will converge to 0. Thus all we need to show is that $\{f(z_n)\}$ will converge to 0. Fix $\epsilon > 0$. Let $N > 1/\epsilon$, and let $\delta = \min\{|p/q - x| : q < N, p \leq q\}$. Let M be such that if $m > M$ then $d(y_m, x) < \delta$. Then note that $f(y_m) < 1/N < \epsilon$, so $d(f(y_m), f(x)) < \epsilon$. Thus $\{f(y_n)\}$ converges to 0, so we see that f is continuous at every irrational x.

(10) 2. We will show that $\mathbf{R}^2 \backslash \{x_1, \ldots, x_n\}$ is path-connected. Let $r = 1/3 \min\{|x_i - x_j| : 1 \leq i, j \leq n\}$. Note that any two circles of radius r centered at two of the removed points do not intersect. Pick any two points in $x, y \in \mathbf{R}^2 \backslash \{x_1, \ldots, x_n\}$. Connect them with a straight line. If this line does not contain any point in $\{x_1, \ldots, x_n\}$ then we are done. Suppose that this line contains some point $x_j \in \{x_1, \ldots, x_n\}$, draw C, the circle of radius r centered around x_j. Then the line will intersect C at two points. Replace the segment inside C with one of the two arcs connecting the two points of intersection. Then this path can clearly be described by a continuous function from $[0, 1]$ into $\mathbf{R}^2 \backslash \{x_1, \ldots, x_n\}$. Thus any two points are connected, so $\mathbf{R}^2 \backslash \{x_1, \ldots, x_n\}$ is connected.

(10) 4. Let f be a homeomorphism $X \to Y$. Note that for any closed set $Z \in X$ f is also a homeomorphism from $X \backslash Z$ to $Y \backslash f(Z)$.

First consider the circle and the interval in $\mathbf{R}$. Suppose that there is some homeomorphism f between them. First note that since homeomorphisms preserve compactness, we know that the interval must be closed, since the circle is compact. Now consider a point x on the interior on the interval. Let $Z = \{x\}$. Then f will be a homeomorphism between two disjoint intervals in $\mathbf{R}$ and a circle with a point removed. However, note that the circle is still connected, and the interval is not. Thus f cannot be a homeomorphism.

Now consider the figure-8 and the circle. Suppose that there is a homeomorphism f between them. Let x be the point of intersection of the figure-8. Then the figure-8 minus x is disconnected, while the circle minus $f(x)$ is still connected. However, f would be a homeomorphism between these two sets, which is a contradiction since homemorphisms preserve conectedness. Thus the circle is not homeomorphic to the figure-8.