# Problem Set #4 Part C
# Official Solutions

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(15) 1.

(2) (1) Note that

$$y_n = \sup_{m \geq n} x_m = \max\{x_n, \sup_{m \geq n+1} x_m\} = \max\{x_n, y_{n+1}\} \geq y_{n+1}.$$

Thus the sequence $\{y_n\}$ is nonincreasing. In addition, we know that there exists $M$ such that $x_n \geq M$ for all $n$. Since $y_n \geq x_n$ by definition, we know that $y_n \geq M$ for all $n$. Thus $\{y_n\}$ is a nonincreasing sequence bounded from below.

(2) (2) Define $z_n = \inf_{m \geq n} x_m$. Let $\liminf x_n = \lim z_n$.

(5) (3) Note that

$$y_n = \sup_{m \geq n} x_n \geq \inf_{m \geq n} x_n = z_n.$$

Note that if we have a convergent sequence where all elements are nonnegative, the limit of the sequence must also be nonnegative. Thus the limit of the sequence $\{y_n - z_n\}$ is nonnegative, so $\limsup x_n \geq \liminf x_n$.

Now suppose that $\limsup x_n = \liminf x_n$. Fix $\epsilon > 0$. Then there will be an $N$ such that for all $n \geq N$ we have $y_n - z_n < \epsilon$. This means that for all $m_1, m_2 > N$ we have $|x_{m_1} - x_{m_2}| < \epsilon$, so $\{x_n\}_{n \geq N}$ is a Cauchy sequence, so it converges, and so has a limit. Note that the limit of any sequence is between lim sup and lim inf of that sequence, so if they are equal the sequence must converge to the lim sup.

Now suppose that $\{x_n\}$ is a convergent sequence, with limit $l$. Fix $\epsilon > 0$. Then there will exist an $N$ such that for all $n \geq N$ $|x_n - l| < \epsilon/2$. Thus for any $m_1, m_2 \geq N$ we will have $|x_n - x_m| \leq |x_n - l| + |l - x_m| < 2\epsilon$. Thus $y_N - z_N \leq \epsilon$. Thus $\lim\{y_n - z_n\} = 0$, so $\limsup x_n = \liminf x_n$. Thus since $\limsup x_n = \liminf x_n$ we see that $\limsup x_n = \lim x_n$, so we are done.

(4) (4) $A(x)$ will be bounded because $\{x_n\}$ is bounded. Then $A(x)$ is compact, so it will contain its largest element.

Fix $\epsilon > 0$. Let $\{x_{\phi(n)}\}$ be a subsequence of $\{x_n\}$ converging to $L$, the largest element of $A(x)$, and suppose that $l < L$. Let $e = (L-l)/2$. Then we can assume without loss of generality that $x_{\phi(n)} > l + e$ for all $n$. Thus $y_n = \sup_{m \geq n} x_n \geq x_{\phi(n)} > l+e$ for all $n$, so $l = \limsup x_n = \lim y_n \geq l+e > l$ which is a contradiction. Thus $l$ is the largest element of $A(x)$.

Note that the number of $n \in \mathbf{N}$ such that $x_n > l + \epsilon$ is finite, since if it were not we would have a subsequence $\{x_{\phi(n)}\}$ such that each element will be larger than $l + \epsilon$, so it will have an accumulation point larger than $l + \epsilon$, which contradicts the fact that $l$ is the largest element.

(2) (5) $\liminf x_n$ will be the smallest element of $A(x)$, $s$. For every $\epsilon > 0$ the set $\{x_n : x_n < s - \epsilon\}$ is finite.

(10) 2. (Sarah Eggleston) First we will show that the sequence is bounded above, so that it is a bounded sequence. Note that

$$n\alpha_n \leq (n-1)\alpha_{n-1} + \alpha_1 \leq (n-2)\alpha_{n-2} + 2\alpha_1 \cdots \leq n\alpha_n$$

so $\alpha_n \leq \alpha_1$. Now let $l = \liminf \alpha_n$. We will show that there exists an $N$ such that for all $n > N$, $|l - \alpha_n| < \delta$. Fix $k$ such that $\alpha_k < l + \delta/2$ (which exists by the definition of $l$).

Let $m > k\alpha_1/\delta$, and define $M \in \mathbf{N}$ to be such that $m$ is between $Mk$ and $(M+1)k$. Then note that $\alpha_{Mk} \leq \alpha_k$ (by the same method we used above). Then

$$\alpha_m \leq \frac{Mk}{m}\alpha_{Mk} + \frac{m - Mk}{m}\alpha_{m-Mk} \leq \frac{Mk}{m}\alpha_k + \frac{m - Mk}{m}\alpha_1.$$

Then from our definition of $M$, we know that $(m - Mk)/m < k/m$. So

$$\alpha_m \leq \alpha_k + (k\alpha_1)/m < l + \delta/2 + \delta/2 = l + \delta.$$

Thus for all $m > k\alpha_1/\delta$, $\alpha_m < l + \delta$. We know that there are finitely many $a$ such that $\alpha_a < l$. Let $A$ be the maximum such $a$, and let $N = \max\{k\alpha_1/delta, A\}$. Then for $n > N$, we know that $|\alpha_n - l| < \delta$. Thus $l$ is the limit of $\{\alpha_n\}$, so we are done.

(15) 3.

(5) (1) Let $s_n = \sum_{i=1}^n a_n$ and let $s'_n = \sum_{i=1}^n a_{\phi(n)}$. Let $A_n = \sum_{i=1}^n |a_n|$. Then we want to show that $\lim s'_n = \lim s_n$. Define $f(n) = \max\{\phi(m) : m \leq n\}$. We know that there exists an $N$ such that for all $n > N$ we have $|A_n - \lim A_n| < \epsilon$. Then for $m \geq f(N)$ we have

$$|s'_m - s_m| = \left| \sum_{\phi(m)>N \text{ or } m>f(N)} (-1)^* a_m \right| \leq \sum_{\phi(m)>N \text{ or } m<f(N)} |a_m| < \sum_{m=N}^{\infty} |a_m| < \epsilon.$$

Thus $\lim(s_n - s'_n) = 0$, so $\lim s_n = \lim s'_n$.

(5) 2. Suppose that either $P$ or $N$ is finite. Without loss of generality, $N$ is finite. Then $\sum_{n\in N} a_n = -M$ for some $M$. Since $\sum a_n$ is convergent, we know that $\sum_{n>\max N} a_n$ converges. It is equal to $\sum_{n>\max N} |a_n|$. But then $\sum |a_n| = \sum_{n<\max N} |a_N| + \sum_{n>\max N} |a_n|$ which is a finite sum plus a convergent sum, so it converges. This is a contradiction. So both $P$ and $N$ are infinite.

First note that if one of $\sum_{p\in P} a_p$ or $\sum_{n\in N}$ diverges then they both must, since if only one did then the total sum would diverge, a contradiction. Now suppose that both of these sums converge to $A$ and $B$, respectively. Note that each of these sums must converge absolutely. Then

$$\sum_{n=1}^{k} |a_n| = \sum_{n\in N, n\le k} |a_n| + \sum_{n\in P, n\le k} |a_n| = \left|\sum_{n\in N, n\le k} a_n\right| + \left|\sum_{n\in P, n\le k} a_n\right| \le |A| + |B|$$

so $\sum a_n$ is absolutely convergent, a contradiction. So each of $\sum_{n\in N} a_n$ and $\sum_{n\in P} a_n$ must diverge.

(5) (3-4) Since $\sum a_n$ is convergent, we know that $|a_n| \to 0$ as $n \to \infty$. Let $N$ be such that $|a_n| < \epsilon$ for all $n > N$, and that for some $k_1, k_2 < N$ $\sum_{i=1}^{k_1} a_{\phi(i)} > z$ and $\sum_{i=1}^{k_2} a_{phi(i)} < z$. Let $M$ be such that $\phi(m) > N$ for all $m > M$. Then we know that for all $m > M$, if $\sum_{i=1}^{m} a_{\phi(i)} > z$ then $\sum_{i=1}^{m} a_{\phi(i)} - z < |a_{\phi(m)}| < \epsilon$, so we see that $\limsup \sum_{i=1}^{m} a_{\phi(i)} \le z$. With a similar argument, we see that $\liminf \sum_{i=1}^{n} a_{\phi(i)} \ge z$. Thus we see that $\limsup \sum_{i=1}^{n} a_{\phi(i)} = \liminf \sum_{i=1}^{n} a_{\phi(i)} = z$ so the sum converges to $z$.