Solution Set 7: Part A

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Mathematics 25a
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November 7, 2003

A.1. *Prove that the function*

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x^2 \sin(1/x) & \text{if } x \neq 0 \end{cases}$$

is differentiable everywhere on $\mathbb{R}$ *but that* f *is not* C^1.

Solution. x^2, $\sin x$, and $1/x$ are all C^1 on $\mathbb{R} \setminus \{0\}$, so f is C^1 on $\mathbb{R} \setminus \{0\}$ with derivative given by

$$f'(x) = 2x \sin(1/x) - \cos(1/x)$$

Clearly, f is C^0 (although this isn't really necessary), since

$$\lim_{x\to 0} |f(x)| = \lim_{x\to 0} \left| x^2 \sin(1/x) \right| \leq \lim_{x \to 0} x^2 = 0 = f(0)$$

We claim that the derivative $f'(0)$ of f at 0 exists and that $f'(0) = 0$. By similar reasoning,

$$\lim_{h\to 0} |h \sin(1/h)| \leq \lim_{h \to 0} h = 0$$

and setting $\phi(h) = h \sin(1/h)$, we may write

$$f(h) = h\phi(h) = f(0) + h\phi(h) \Rightarrow f'(0) = 0$$

since $\phi(h) \to 0$. Thus, f is differentiable everywhere. However,

$$\lim_{x\to 0} f'(x) = \lim_{x \to 0} (2x \sin(1/x) - \cos(1/x)) = -\lim_{x\to 0} \cos(1/x)$$

does not exist, and therefore $f'(x)$ is not continuous on $\mathbb{R}$. To see this explicitly, we could look at the sequence $x_n = (2\pi n)^{-1} \to 0$; if f' were continuous, we would have $f'(x_n) \to f'(0) = 0$, but for all n

$$f'(x_n) = \frac{1}{\pi n} \sin(2\pi n) - \cos(2\pi n) = -1$$

so f' is not continuous at 0. The fact that $\lim_{x\to 0} f'(x)$ doesn't exist means that we couldn't make f' continuous even by redefining f' at 0 (to be -1, for instance). □

A.2. *Suppose that* $\{f_n\}_{n\geq 1}$ *is a sequence of* C^1 *functions* $f_n : [0, 1] \to \mathbb{R}$, *and that* $f, g : [0, 1] \to \mathbb{R}$ *are two functions such that:* $f_n \to f$ *pointwise, and* $f_n' \to g$ *uniformly on* $[0, 1]$. *Prove that* f *is* C^1, *that* $f' = g$, *and that* $f_n \to f$ *uniformly.*

Solution. (due to Allan Chu) Note that g is continuous since it is the uniform limit of continuous functions f_n'. Indeed, for any $\epsilon > 0$, there exist N such that for all $n > N$,

$$|f_n'(x) - g(x)| < \frac{\epsilon}{3} \text{ and } |f_n'(y) - g(y)| < \frac{\epsilon}{3}$$

For any particular $n > N$, $\exists \delta$ such that for all $x, y \in [0, 1]$ with $|x - y| < \delta$,

$$|f_n'(x) - f_n'(y)| < \frac{\epsilon}{3}$$

$$\Rightarrow |g(x) - g(y)| \leq |g(x) - f_n'(x)| + |f_n'(x) - f_n'(y)| + |f_n'(y) - g(y)| < \epsilon$$

Given $\epsilon > 0$, since $f_n' \to g$ uniformly, there exists N such that for all $n > N$, $x \in [0, 1]$

$$\left| \int_0^x (f_n' - g) \right| \leq \int_0^x |f_n' - g| \leq \int_0^x \epsilon \leq \epsilon$$

f_n is C^1, so $\int_0^x f_n' = f_n(x) - f_n(0)$, and the above says that $f_n(x) - f_n(0)$ converges uniformly to $G(x) = \int_0^x g$. But we already know that $f_n(x) - f_n(0)$ converges pointwise to $f(x) - f(0)$, so $f(x) - f(0) = G(x)$. $f_n(x) - f_n(0)$ converges to $f(x) - f(0)$ uniformly and $f_n(0) \to f(0) \Rightarrow f_n \to f$ uniformly. g is continuous, so by the fundamental theorem of calculus $G(x) = f(x) - f(0)$ is C^1, and therefore f is C^1 with derivative $f' = G' = g$. $\square$