PROBLEM SET #9 PART B
OFFICIAL SOLUTIONS
TOTAL POINTS: 40

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(5) B1. We know that

$$\sum_{k=0}^{n} e^{ikx} = \frac{1 - e^{i(n+1)z}}{1 - e^{ix}}.$$

At the same time, we know that $e^{i\theta} = \cos\theta + i\sin\theta$. Thus we see that

$$\sum_{k=0}^{n} \cos kx = \operatorname{Re} \sum_{k=0}^{n} e^{ikx} \qquad \sum_{k=0}^{n} \sin kx = \operatorname{Im} \sum_{k=0}^{n} e^{ikx}.$$

Plugging in that $e^{i\theta} = \cos\theta + i\sin\theta$ and simplifying we see that

$$\frac{1 - e^{i(n+1)z}}{1 - e^{ix}} = \frac{1 - \cos x - \cos(n+1)x + \cos nx}{2 - 2\cos x} + i\frac{\sin x - \sin(n+1)x + \sin nx}{2 - 2\cos x}.$$

Thus

$$\begin{aligned} \sum_{k=0}^{n} \cos kx &= \frac{1 - \cos x - \cos(n+1)x - \cos(n+2)x}{2 - 2\cos x} \\ \sum_{k=0}^{n} \sin kx &= \frac{\sin x + \sin(n+1)x - \sin(n+2)x}{2 - 2\cos x}. \end{aligned}$$

(5) B2. Let P be of degree n. Let $P(z_0) = re^{i\theta}$, and write

$$P(z) = re^{i\theta} + b_1(z - z_0) + \cdots + b_n(z - z_0)^n.$$

(We can do this because P is a polynomial.) Let j be the minimum index such that $b_j \neq 0$. For $0 < \delta$, let

$$z = z_0 + \frac{\delta}{b_j^{1/j}} e^{\theta i/j}.$$

Then

$$P(z) = re^{i\theta} + b_j \frac{\delta^j}{b_j} e^{i\theta} + O(\delta^{j+1})$$

which will have bigger absolute value than $P(z_0)$. Thus for all $\epsilon > 0$ we can find a point z such that $|z - z_0| < \epsilon$ but $|P(z)| > |P(z_0)|$ so P has no local maximum.

(15) B3.

(5) (1) Suppose that P is split with n distinct roots $\alpha_1, \ldots, \alpha_n$ ($\alpha_1 < \cdots < \alpha_n$) with multiplicities $\mu_1, \ldots, \mu_n$. Note that, by the mean value theorem, P' will have a zero between α_i and α_{i+1}. Also, since $P(x) = (x - \alpha_i)^{\mu_i} Q(x)$ we know that $P'(x) = \mu_i (x - \alpha_i)^{\mu_i - 1} Q(x) + (x - \alpha_i)^{\mu_i} Q'(x)$, so $P'(x)$ will have a 0 of multiplicity $\mu_i - 1$ at each α_i (where a zero of multiplicity 0 is not a zero). Thus we have found

$$\sum_{j=1}^{n} (\mu_i - 1) + (n - 1) = d - 1$$

zeroes of P'. Since $\deg P' = d - 1$ we have found all zeroes, so all zeroes are real. Therefore P' is split.

(5) (2) Let n, α_i, μ_i be as in the previous part. Also, note that $P' + \lambda P$ will have a zero of multiplicity $\mu_i - 1$ at each α_i. Notice that $P(x)e^{\lambda x}$ will have zeroes at each α_i; therefore its derivative must have a zero between α_i and α_{i+1}. Since $e^{\lambda x} \neq 0$ for all $x \in \mathbf{R}$, and

$$(P(x)e^{\lambda x})' = (\lambda P(x) + P'(x))e^{\lambda x}$$

we see that $\lambda P + P'$ must have a zero between α_i and α_{i+1}. Thus we have found $d-1$ zeroes for this polynomial. Note, however, that as $x \to -\infty$ $P(x)e^{\lambda x} \to 0$. Since it is not uniformly 0 below α_1 there must be some 0 of the derivative less than α_1. Therefore we have found d real zeroes for a polynomial of degree d, so we have found them all.

(5) (3) We will prove this by induction on d. In the previous part we showed the base case. Now suppose that the desired conclusion holds for $d-1$. Then let $Q = b_0 + b_1 x + \cdots + b_d x^d$ be split. Then we know that $Q = (x - \alpha)(c_0 + c_1 x + \cdots + c_{d-1}x^{d-1})$ for some polynomial of degree $d-1$. Let

$$R = c_0 P(x) + c_1 P'(x) + \cdots + c_{d-1} P^{(d-1)}(x).$$

Note that

$$R - \alpha R' = b_0 P(x) + b_1 P'(x) + \cdots + b_d' P^{(d)}(x).$$

We know that R is split, by the inductive hypothesis. Therefore $R - \alpha R'$ is split, so the desired conclusion holds.

(15) B4.

(7) (1) We will first prove, by induction, that the convex hull of n points is the smallest convex polygon that contains all of the points. For two points, a and b we have that the convex hull is the set of $\lambda a + (1 - \lambda b)$ for $\lambda \in [0, 1]$. However, note that this is $b + \lambda(a - b)$ which is clearly the segment between a and b. Now suppose that the statement holds for $n-1$ points. Consider the case with n points, $a_1, \ldots, a_n$. Fix a nonnegative λ_n. Note that for any nonnegative $\lambda_1, \ldots, \lambda_{n-1}$ such that their sum is less than or equal to 1,

$$\lambda_1 a_1 + \cdots + \lambda_{n-1} a_{n-1} = \lambda(t_1 a_1 + \cdots + t_{n-1} a_{n-1})$$

where

$$\lambda = \lambda_1 + \cdots + \lambda_{n-1} \text{ and } t_i = \lambda_i / \lambda.$$

Note that $t_1 a_1 + \cdots + t_{n-1} a_{n-1}$ is a point in the convex hull of the first $n-1$ points. Then we know that $\lambda_n = 1 - \lambda$. So the point obtained by these lambdas will be on the segment between $t_1 a_1 + \cdots + t_{n-1} a_{n-1}$ and a_n. Therefore it will be contained in sone triangle with one vertex on an a_n and the other two among the a_i. Therefore the convex hull will be contained in the smallest convex polygon containing all a_i.

Now consider the smallest convex polygon containing all a_i. Clearly the sides of this polygon will be contained in the convex hull of the a_i since we know that a segment between two points is contained in the convex hull. Now consider any other point inside the polygon. Draw a line through the point and any vertex (WLOG a_1) on the sides of the polygon. This line will intersect the polygon in one more point, x. We know, by a similar argument to before, that the segment $a_1 x$ is contained in the convex hull. Thus all points inside the polygon are contained in the convex hull.

Thus the convex hull of n points is the smallest convex polygon containing the points. Notice that this can be written as the union of the interiors of all triangles with the a_i as vertices.

We know that the roots of $x^n - 1$ are a regular n-gon centered at the origin with one vertex on 1. Thus the convex hull of these roots is the n-gon including its interior. Thus for $n = 2$ it is the segment from 1 to -1, for $n = 3$ it is the equilateral triangle, for $n = 4$ it is the square, and for $n = 5$ it is the regular pentagon. All of these are centered at the origin and have one vertex at 1.

(3) (2) Let $P = (z - z_1) \cdots (z - z_d)$. Then

$$P' = \sum_{j=1}^{d} (z - z_1) \cdots (z - z_{j-1})(z - z_{j+1}) \cdots (z - z_d) = \sum_{j=1}^{d} \frac{P(z)}{z - z_i}.$$

Thus

$$\frac{P'(z)}{P(z)} = \sum_{j=1}^{d} \frac{1}{z - z_i}.$$

(5) (3) **Solution 1**: Consider a point z_0 that is outside of the convex hull of the z_i. Then we know that there is some line L through z_0 such that all of the z_i are on the same side of the line. Then the vectors $z_i - z_0$ will all point to one side of this line (and not along it). Therefore the vectors $1/(z_i - z_0)$ will all point to one side of $\overline{L}$, the reflection of L over the real axis. However, this is a contradiction since these vectors cannot add to 0, since they all contain a positive component along the line perpendicular to $\overline{L}$. Therefore z_0 is not a 0 of $\sum 1/(z - z_i)$ so it is not a zero of $P'(z)$. Thus all zeroes of $P'(z)$ lie inside the convex hull.

Solution 2: Suppose that w is a zero of P'. Let $l_i = 1/|w - z_i|^2$, and let $l = \sum_{j=1}^{d} l_j$. Then

$$\sum_{j=1}^{d} \frac{1}{w - z_j} = 0 \Longrightarrow \sum_{j=1}^{d} \frac{\overline{w}}{|w - z_j|^2} = \sum_{j=1}^{d} \frac{\overline{z}_j}{|w - z_j|^2}.$$

Thus we have that

$$\overline{w}l = \sum_{j=1}^{d} z_j l_i \Longrightarrow \overline{w} = \sum_{j=1}^{d} \frac{l_j}{l} \overline{z}_j \Longrightarrow w = \sum_{j=1}^{d} \frac{l_j}{l} z_j.$$

Clearly, each l_i is nonnegative. In addition, $\sum l_i/l = 1$. Thus this point will be in the convex hull of the z_i.