# Solution Set 11: Part A

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**A.1. Matrices of trace 0 I.** *Let $M \in M_n(K)$ be a (nonzero) matrix whose trace is 0.*

(1) *Prove that there exists an invertible matrix $P$ such that the first column of $P^{-1}MP$ is*

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

*Solution.* Let $e_1, \cdots, e_n$ be the standard basis for $K^n$ (i.e., $e_i$ has a 1 in the $i$th position and zeroes everywhere else) and let $f : K^n \to K^n$ be the map represented by $M$ in the basis $e_i$. We must find another basis $v_i$ such that $v_2 = f(v_1)$; it will be sufficient to find a $v$ that isn't an eigenvector (i.e., $f(v), v$ is free). If every $v \in K^n$ is an eigenvector, then for every $i, j$, if $\lambda_i$ is the eigenvalue of $e_i$, there must exist $\lambda \in K$ with

$$\lambda(e_i + e_j) = f(e_i + e_j) = \lambda_i e_i + \lambda_j e_j \;\Rightarrow\; \lambda = \lambda_i = \lambda_j$$

since $e_i$ is a basis; thus, $f = \lambda \mathrm{Id}$. But $0 = \mathrm{Tr}(f) = n\lambda \Rightarrow \lambda = 0$, and $f = 0$, contrary to hypothesis. Thus, such a $v$ must exist.

(2) *Prove that there exists an invertible matrix $Q$ such that all the diagonal entries of $Q^{-1}MQ$ are 0.*

*Solution.* This follows from applying (1) $n$ times. After applying (1) once, we have a matrix of the form

$$\left(\begin{array}{c|cccc} 0 & * & * & * & * \\ \hline 1 & & & & \\ 0 & & & & \\ \vdots & & A & & \\ 0 & & & & \end{array}\right)$$

where $A$ is an $n-1 \times n-1$ matrix; The trace is independent of basis, so because the first diagonal entry is 0, $\mathrm{Tr}(A) = 0$. Now apply (1) repeatedly until we obtain the desired form.

**A.2. Matrices of trace 0 II.** *Let*

$$D = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & n \end{pmatrix}$$

*and let $\phi : M_n(K) \to M_N(K)$ be the map defined by $\phi(M) = MD - DM$.*

(1) *Prove that* $\ker \phi$ *is the set of all diagonal matrices. What is the rank of $\phi$?*

*Solution.* Multiplying some $M \in M_n(K)$ on the right by $D$ will multiply the $j$ the column by $j$, while multiplying by $D$ on the left will multiply the $i$th row by $i$. Thus, $\phi(M)_{ij} = (j-i)M_{ij}$, and $\phi(M) = 0$ if and only if $M_{ij} = 0$ for all $i \neq j$, i.e., $M$ is diagonal.

$M_n(K)$ is a $K$-vector space of dimension $n^2$ (with basis the matrices $E_{i,j}$ in A.4.); the space of diagonal matrices has dimension $n$ (with basis $E_{i,i}$). Thus, $\mathrm{rk}(\phi) = \dim M_n(K) - \dim\ker(\phi) = n^2 - n = n(n-1)$.

(2) *Prove that* $\mathrm{im}(\phi)$ *is a subspace of the space of matrices whose diagonal entries are 0.*

*Solution.* This follows immediately from the above formula for $\phi(M)$.

(3) *Prove that* $\mathrm{im}(\phi)$ *is the space of matrices whose diagonal entries are 0.*

*Solution.* The space of matrices with zeroes along the diagonal has dimension $n^2 - n = n(n-1) = \mathrm{rk}(\phi)$, so the two must in fact be equal.

**A.3. Matrices of trace 0 III (Shoda's theorem).** *Using the previous two problems, prove that a matrix $M$ satisfies* $\mathrm{Tr}(M) = 0$ *if and only if there exist $A, B \in M_n(K)$ such that $AB - BA$.*

*Solution.* Clearly $0 \in \mathrm{im}(\phi)$. If $M \neq 0$ has trace 0, then by A.1.1 there exists an invertible matrix $Q$ such that $Q^{-1}MQ \in \mathrm{im}(\phi)$, by A.2.3, so there is matrix $C$ such that $Q^{-1}MQ = CD - DC$. Thus,

$$M = QCDQ^{-1} - QDCQ^{-1} = (QCQ^{-1})(QDQ^{-1}) - (QDQ^{-1})(QCQ^{-1})$$

Conversely, every matrix in $\mathrm{im}(\phi)$ is traceless.

**A.4. Matrices of trace 0 IV.** *Prove that the subspace of $M_n(K)$ generated by all nilpotent matrices is the space of matrices whose trace is 0.*

*Solution.* Let $T$ be the space of traceless matrices. We know that a matrix $M$ is nilpotent if and only if there is an invertible matrix $P$ such that $P^{-1}MP$ is upper triangular with zeroes along the diagonal. The trace is independent of basis, so $M \in T$. Conversely, for any $M \in T$, there exists an invertible $P$ such that $Q^{-1}MQ$ has 0's along the diagonal. Take $A$ to be the lower-triangular part of $Q^{-1}MQ$, and $B$ the upper triangular part, both with zeroes along the diagonal. $A, B$ are thus both nilpotent, and therefore so are $QAQ^{-1}, QBQ^{-1}$; we then have $M = QAQ^{-1} + QBQ^{-1}$.

**A.5. Skolem-Noether's theorem.** *Let $\phi : M_n(K) \to M_n(K)$ be a bijective map such that $\phi(A + \lambda B) = \phi(A) + \lambda\phi(B)$, $\phi(\mathrm{Id}) = \mathrm{Id}$, and $\phi(AB) = \phi(A)\phi(B)$. Such a map is called an automorphism of $M_n(K)$. The goal of this problem is to find all automorphisms of $M_n(K)$.*

(1) *Check that if $P$ is an invertible matrix, then the map $\phi_P$ defined by $\phi_P(M) = PMP^{-1}$ is an automorphism of $M_n(K)$.*

*Solution.* The inverse of $\phi_P$ is $\phi_{P^{-1}}$, so $\phi_P$ is bijective. Since multiplication is distributive, $\phi_P(A + \lambda B) = P(A + \lambda B)P^{-1} = PAP^{-1} + \lambda PBP^{-1} = \phi_P(A) + \lambda\phi_P(B)$. Clearly $\phi_P(\mathrm{Id}) = P\mathrm{Id}P^{-1} = \mathrm{Id}$, and $\phi_P(AB) = PABP^{-1} = (PAP^{-1})(PBP^{-1}) = \phi_P(A)\phi_P(B)$.

(2) *Let $E_{i,j}$ be the matrix all of whose entries are 0 except the $(i,j)$-th which is 1. Compute $E_{i,j}E_{k,\ell}$ (the answer depends on whether $j = k$ or not).*

*Solution.* After a little thought, its pretty obvious that $E_{i,j}E_{k,\ell} = E_{i,\ell}$ if $j = k$ and 0 otherwise.

(3) *Now let $\phi$ be an unspecified automorphism and let $U_{i,j} = \phi(E_{i,j})$. Prove that* $\ker(U_{1,1} - \mathrm{Id})$ *is non-trivial.*

*Solution.* Note that $\phi(0) = 0$, by linearity. We have $E_{1,1}E_{1,1} = E_{1,1}$ by (2), so $(U_{1,1} - \mathrm{Id})U_{1,1} = 0$. $\phi$ is bijective, so $U_{1,1} \neq 0$, and $\mathrm{im}(U_{1,1}) \neq 0$, implying that $\ker(U_{1,1} - \mathrm{Id}) \supset \mathrm{im}(U_{1,1})$ is nontrivial.

(4) *Let $f_1 \in \ker(U_{1,1} - \mathrm{Id})$ be a nontrivial element and for $k \geq 2$ set $f_k = U_{k,1}(f_1)$. Prove that $\{f_1, \ldots, f_n\}$ is a basis of $K^n$.*

*Solution.* Consider the matrix

$$T = U_{1,n} + U_{n,n-1} + \cdots + U_{2,1} = \phi(A)$$

where

$$A = E_{1,n} + E_{n,n-1} + \cdots + E_{2,1} = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{pmatrix}$$

$A$ is invertible, so $T$ is as well, and therefore $f_k = T^{k-1} f_1$ is nonzero for each $k$. Note that $U_{k,k} f_i$ is 0 for $i \neq k$ and $f_i$ for $i = k$. Suppose we have

$$0 = \lambda_1 f_1 + \cdots + \lambda_n f_n$$

Applying $U_{k,k}$, we find $\lambda_k f_k = 0$, and so by the above $\lambda_k = 0$. Thus, the $f_1, \ldots, f_n$ are free. There are $n$ of them, so they form a basis.

(5) *Let $P$ be the matrix of $f_i$ in the usual basis of $K^n$. prove that $\phi$ and $\phi_P$ coincide on the $E_{k,1}$'s.*

*Solution.* Consider $E_{k,1}$ as representing a map in the $f_i$ basis. Thus, $E_{k,1} f_i$ is 0 for $k \neq i$ and $f_i$ for $i = k$. As we saw above, $U_{k,1}$ represents the same map but in the usual basis. The relations between them is

$$\phi_P(E_{k,1}) = P E_{k,1} P^{-1} = U_{k,1} = \phi(E_{k,1})$$

(6) *Prove by recurrence on $n$ that if $\phi$ is an automorphism of $M_n(K)$, then there exists an invertible $Q$ such that $\phi(M) = QMQ^{-1}$.*

*Solution.* We don't really need induction. Let $e_i$ be the standard basis of $K^n$. With the same notation as in (5), we have

$$U_{i,j} f_k = U_{i,j} U_{k,1} f_1 = \begin{cases} e_i & j = k \\ 0 & i \neq j \end{cases}$$

and similarly

$$E_{i,j} e_k = \begin{cases} f_i & j = k \\ 0 & i \neq j \end{cases}$$

Thus, $E_{i,j}$ is the matrix of $U_{i,j}$ in the $f_i$ basis. Therefore, $P E_{i,j} P^{-1} = U_{i,j}$ for all $i, j$, and $\phi = \phi_P$, by linearity.