# Problem Set #11 Part B
# Official Solutions
# Total Points: 44

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(7) 1.

(5) (1) We construct a basis for $V$ by taking pairs of vectors $v, fv$, so that every vector $v$ in a basis has a $fv$ also in the basis (or has a multiple in the basis). Then the vectors $v$ such that $fv = v$ are eigenvectors with eigenvalue 1. Consider the vectors $v$ such that $fv \neq v$. If $fv = -v$ then $v$ is an eigenvector with eigenvalue $-1$. Now suppose that $v$ is not proportional to $fv$. Then consider the vectors $fv + v$, $fv - v$. These will also form a basis of the space spanned by $v, fv$. $f(fv + v) = f^2v + fv = v + fv$, so it is an eigenvector with eigenvalue 1. $f(fv - v) = f^2v - fv = v - fv$, so it is an eigenvector with eigenvalue $-1$. Thus we have a basis of eigenvectors, so $f$ is diagonalizable, with eigenvalues $\pm 1$.

(2) (2) Such a map is called a symmetry because the subspace $V_-$ is reflected over the subspace $V_+$.

(5) 2. First note that if $V$ is a bijection then $\ker V = \{0\}$ so $V = \ker V \oplus \operatorname{im} V$. Thus we will ignore this case.

Suppose that 0 is not a root of the characteristic polynomial. Then we know that $\Pi_f = fP(f) + c$ for $c \neq 0$. Thus we see that $v = -\frac{1}{c} fP(f)v$, so $v \in \operatorname{im} f$. Thus $f$ is injective, and therefore it is a bijection, which is the case we are not considering. Now suppose that 0 is a root with multiplicity $k > 1$. Then $\Pi_f = f^k P(f)$ with $P(0) \neq 0$. Then we know that $0 = f^k P(f)v$ for all $v$. Thus $f^{k-1}P(f)v$ is both in the kernel and the image, since $f(f^{k-1}P(f)v) = 0$, and $f(f^{k-2}P(f)v) = f^{k-1}P(f)v$. Thus $\ker f \cap \operatorname{im} f \neq \{0\}$, so $V \neq \ker f \oplus \operatorname{im} f$. Thus if $V = \ker f \oplus \operatorname{im} f$ then 0 is a simple root of $\Pi_f$.

Now suppose that 0 is a simple root of $\Pi_f$. Then we know that $\Pi_f = fP(f)$, with $P(0) \neq 0$. Notice that if $P(f)v = 0$ then $v = -\frac{1}{P(0)}(P(f) - P(0))$, which means that $v \in \operatorname{im} f$. On the other hand, if $v \in \ker f$ then $P(f)v \neq 0$, since $P(f)v = P(0)v$. Thus

$\ker f \cap \operatorname{im} f = \{0\}$. Since we already knew that $\dim V = \dim \ker f + \dim \operatorname{im} f$, we see that $V = \ker f \oplus \operatorname{im} f$.

(10) 3.

(4) (1) Consider the minimal polynomial for $f$. Suppose that it is linear; this means that $f$ is a multilple of the identity. In this case $\ker f = \{0\}$ and $\operatorname{im} f = V$, so $V = \ker f \oplus \operatorname{im} f$ and we are done.

Now suppose that the minimal polynomial is not linear. We know that the polynomial $f^2 - f$ works, so we see that it is the minimal polynomial. Since 0 is a simple root of $x^2 - x$ we see, by the previous problem, that $V = \operatorname{im} f \oplus \ker f$.

(4) (2) Notice that, for all $v$ such that $v = fw$ (for some $w$), $fv = v$. Thus on the image of $f$, $f$ is the identity and so has eigenvalues 1. On the kernel it has eigenvalues 0. From the first part we see that we can have a basis which consists only of vectors in the image or the kernel, so we see that $f$ is diagonalizable, and has eigenvalues 1 and 0.

(2) (3) $f$ is called a projection because it kills takes a splitting of $V$ as $V_1 \oplus V_2$ and kills the vectors that are in $V_1$, "projecting" all vectors onto their component in $V_2$.

(5) 4. Consider a polynomial of degree $m$. We will show that for any $m \geq 0$ we can find a polynomial $P(x)$ such that $f(P(x)) = m(m-1)P(x)$.

For $m = 0, 1$ this is obvious since $f$ kills all polynomials of degrees $0, 1$, so the polynomials $1, x$ work. Suppose $m > 1$. Then, if $P(x) = \sum a_n x^n$ we know that $f(P(x)) = (x^2 - 1)\sum n(n-1)a_n x^{n-2} = \sum((n-2)(n-3)a_{n-2} - n(n-1)a_n)x^{n-2}$. Notice that for $n < m - 1$ this means that we can express $a_n$ in terms of $a_{n+2}$. If we set $a_m = 1$ and $a_{m-1} = 0$ and then let $a_n = \frac{(n+1)(n+1)}{n(n-1)-m(m-1)}a_{n+2}$ we a polynomial with the desired characteristics. Since this worked for every degree we see that the map is diagonalizable, since we can take an eigenvector of every degree and this will form a basis.

(5) 5. Notice that

$$M^k = \begin{pmatrix} A^k & 0 \\ 0 & \\ B^k & \end{pmatrix}.$$

Therefore for any polynomial $P$

$$P(M) = \begin{pmatrix} P(A) & 0 \\ 0 & \\ P(B) & \end{pmatrix}.$$

We see that $P(M) = 0$ if and only if $P(A) = 0$ and $P(B) = 0$. Therefore $\Pi_A | \Pi_M$ and $\Pi_B | \Pi_M$. The polynomial of least degree such that this holds is $\operatorname{lcm}(\Pi_A, \Pi_B)$, so we see that $\Pi_M = \operatorname{lcm}(\Pi_A, \Pi_B)$.

(7) 6.

(2) (1) We will prove this by induction. The base case is true by definition. Suppose that $M^kN - NM^k = k\lambda M^k$. Then

$$\begin{aligned} k\lambda M^{k+1} &= M(M^kN - NM^k) = M^{k+1}N + (MN)M^k \\ &= M^{k+1}N - (MN - \lambda M)M^k = M^{k+1}N - NM^{k+1} - \lambda M \end{aligned}$$

which implies the desired result.

(5) (2) Notice that if $\pi_M = \sum_{i=0}^n a_i x^i$ we have

$$0 = \Pi_M N - N\Pi_M = \lambda \sum_{i=0}^n a_i i x^i.$$

The polynomial on the right has the same degree as $\Pi_M$, so it must be an integer multiple of it. It is not unless all $a_i = 0$ for $i < n$. Thus we see that $\Pi_M = x^n$, which means that $M$ is nilpotent.