Math 25a/55a – Honors Advanced Calculus and Linear Algebra
Solutions to Problem Set B

1. (Rudin, page 24, 7) (a) First, I will show that $\bigcup_{i=1}^{n} \bar{A}_i \subset \bar{B}$. To this end, let x be an arbitrary element of $\bigcup_{i=1}^{n} \bar{A}_i$. From the definition of a union, $x \in \bar{A}_i$ for some i. That is, x is in the closure of A_i, so there is a sequence of points $x_n \in A_i$ such that $x_n \to x$. But each point x_n is in $A_i \subset \bigcup_{i=1}^{n} A_i = B$. So $\{x_n\} \subset B$, so x is a limit of a sequence in B. That is, $x \in \bar{B}$. We have shown that any element x of $\bigcup_{i=1}^{n} \bar{A}_i$ is also in $\bar{B}$. That is, $\bigcup_{i=1}^{n} \bar{A}_i \subset \bar{B}$.

Now, I want to show the other inclusion, namely $\bar{B} \subset \bigcup_{i=1}^{n} \bar{A}_i$. So, let $x \in \bar{B}$. There is a sequence of points $x_n \in B = \bigcup_{i=1}^{n} A_i$ such that $x_n \to x$. Now, let's use the fact that B is defined as the union of finitely many sets: There has to be at least one A_i containing x_n for infinitely many values of n. The reason is that, for each of the infinitely many n, x_n has to be in some A_i. If each A_i contained x_n for only finitely many values of n, there would be only finitely many values of n with x_n in *any* of the A_i's. There would be infinitely many n's left over, with no A_i to belong to.

So, let's suppose that some particular A_i contains infinitely many x_n. We can form a subsequence of x_n consisting of just the points in A_i. This subsequence converges to x just as readily as the whole sequence, so x is the limit of a sequence in A_i. That is, $x \in \bar{A}_i$, so $x \in \bigcup_{i=1}^{n} \bar{A}_i$. We have shown that $\bar{B} \subset \bigcup_{i=1}^{n} \bar{A}_i$, so combining this with the other inclusion, $\bar{B} = \bigcup_{i=1}^{n} \bar{A}_i$, and we're done.

(b) The proof that $\bigcup_{i=1}^{\infty} \bar{A}_i \subset \bar{B}$ works the same as the proof in (a) that $\bigcup_{i=1}^{n} \bar{A}_i \subset \bar{B}$. Just replace every "$n$" with an "$\infty$".

To show that it is possible that $\bigcup_{i=1}^{\infty} \bar{A}_i \neq \bar{B}$, consider $A_i = \{\frac{1}{i}, \frac{1}{i-1}, \ldots, 1\} = \{\frac{1}{n}\}_{n=1}^{i}$. We can show $\bar{A}_i = A_i$; for instance, use the fact that a single-point set is closed, and that the union of finitely many closed sets is closed. (It is easy to see from the either definition of limit point that a single-point set $\{x\}$ can have no limit points. Therefore, it is a true statement that $\{x\}$ contains all its limit points. Therefore, $\{x\}$ is closed.) So $\bigcup_{i=1}^{\infty} \bar{A}_i = \bigcup_{i=1}^{\infty} A_i$. Now, $B = \bigcup_{i=1}^{\infty} A_i = \{\frac{1}{n}\}_{n=1}^{\infty}$, so the seqence $\frac{1}{n}$, which converges to 0, is in B; therefore $0 \in \bar{B}$. But $0 \notin B = \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \bar{A}_i$, so we have found a point which is in $\bar{B}$ but not $\bigcup_{i=1}^{\infty} \bar{A}_i$, so $\bigcup_{i=1}^{\infty} \bar{A}_i \neq \bar{B}$.

2. (a) Suppose $\{x_n\} \in l_1$, so $\sum_1^\infty |x_n|$ is finite. (This is also expressed by saying that the sum converges.) We will show that $\sum_1^\infty (x_n)^2$ is finite, so $\{x_n\} \in l_2$. First, we need a lemma.

Lemma. If $\sum_1^\infty a_n$ converges, then $a_n \to 0$ as $n \to \infty$.

Proof. By hypothesis, $s_N = \sum_1^N a_n$ converges to some limit (the sum of the series) as $N \to \infty$. Any convergent sequence is a Cauchy sequence; that is, for any $\epsilon > 0$ there is an integer n_0 such that once $N > n_0$ and $M > n_0$, $d(s_N, s_M) < \epsilon$. In particular, since the last part is true for any $M > n_0$, it is true for $M = N + 1$, so $\epsilon > d(s_N, s_{N+1}) = |s_N - s_{N-1}| = |\sum_1^N a_n - \sum_1^{N-1} a_n| = |a_N|$. (Note that I can replace d with absolute values only because we are working in $\mathbf{R}$, with the standard metric.) I have shown that for any $\epsilon > 0$, there is an integer n_0 such that once $N > n_0$, $d(a_N, 0) = |a_N| < \epsilon$. In other words, $a_N \to 0$. QED.

Applying the Lemma, we see that $|x_N| \to 0$. Therefore, there is an integer n_0 such that whenever $N > n_0$, we have $|x_N| < 1$. (I got this by taking $\epsilon = 1$ in the definition of limit.) For $N > n_0$, we have $(x_n)^2 \leq |x_n|$. That is, after we pass n_0, every term of $\sum_1^\infty (x_n)^2$ is less than or equal to the corresponding term of $\sum_1^\infty |x_n|$, so the sum of any chunk of terms $(x_n)^2$ is less than or equal to the sum of the corresponding chunk of $|x_n|$'s, which is less than or equal to $\sum_1^\infty |x_n|$. So, for $N > n_0$, $\sum_1^N (x_n)^2 = \sum_1^{n_0}(x_n)^2 + \sum_{n_0+1}^N (x_n)^2 \le \sum_1^{n_0}(x_n)^2 + \sum_1^\infty |x_n|$. This last expression is a finite constant, independant of N, so the limit as $N \to \infty$ of the partial sums must also be $\le \sum_1^{n_0}(x_n)^2 + \sum_1^\infty |x_n|$, so this limit (i.e., the value of $\sum_1^\infty (x_n)^2$ is finite. (More rigorously, this is an upper bound on an increasing sequence, which converges to a (finite) value.) Therefore $\{x_n\} \in l_2$. We have shown that $l_1 \subset l_2$.

The next inclusion is easier. If $\{x_n\} \in l_2$, then $\sum_1^\infty (x_n)^2$ converges, so by our Lemma, $(x_n)^2 \to 0$. A convergent sequence is bounded, so $(x_n)^2$ is bounded, so x_n is bounded, so $\{x_n\} \in l_\infty$. We have shown that $l_2 \subset l_\infty$.

If $\{x_n\} \in l_1$, then $\sum_1^\infty |x_n|$ is finite, and so is $|a|\sum_1^\infty |x_n| = \sum_1^\infty |ax_n|$. (This step needs to be checked in terms of the definition of the sum of a series as a limit of partial sums, but it's not hard.) Therefore, $\{ax_n\} \in l_1$. Similar results can be proved for l_2 and l_∞, using $\sum_1^\infty (ax_n)^2 = a^2 \sum_1^\infty (x_n)^2$, and $|x_n| \le M \Rightarrow |ax_n| \le |a|M$ (i.e., $|a|M$ is a bound for $\{ax_n\}$).

Now let's consider the series $\{x_n + y_n\}$. For l_1, $\sum_1^N |x_n + y_n| \le \sum_1^N |x_n| + \sum_1^N |y_n| \le \sum_1^\infty |x_n| + \sum_1^\infty |y_n|$, which is a (finite) upper bound on an increasing sequence of partial sums, which converges. Therefore, $\{x_n + y_n\} \in l_1$, provided $\{x_n\}$ and $\{y_n\}$ are also in l_1. Furthermore, since we have an bound on the sequence $\{\sum_1^N |x_n + y_n|\}$, we also know that its limit (i.e., $\sum_1^\infty |x_n + y_n|$) is less than or equal to that bound. (The limit of a sequence in $(-\infty, A]$ must also be in $(-\infty, A]$ since this interval is a closed set.) In other words, $\sum_1^\infty |x_n + y_n| \le \sum_1^\infty |x_n| + \sum_1^\infty |y_n|$. We'll need this for the triangle inequality.

In l_2, we see that $(\sum_1^N (x_n + y_n)^2)^{1/2} \le (\sum_1^N (x_n)^2)^{1/2} + (\sum_1^N (y_n)^2)^{1/2}$ is true because of the triangle inequality for d_2 on $\mathbf{R}^N$ (consider the points 0, $\{x_n\}_{N=1}^N$, and $\{x_n + y_n\}_{n=1}^N$). Squaring both sides, $\sum_1^N (x_n + y_n)^2 \le \sum_1^N (x_n)^2 + \sum_1^N (y_n)^2 + 2(\sum_1^N (x_n)^2)^{1/2}(\sum_1^N (y_n)^2)^{1/2}$. Using the same technique as in l_1 to let N go to infinity, we see $\sum_1^\infty (x_n + y_n)^2 \le \sum_1^\infty (x_n)^2 + \sum_1^\infty (y_n)^2 + 2(\sum_1^\infty (x_n)^2)^{1/2}(\sum_1^\infty (y_n)^2)^{1/2}$. In particular, $\sum_1^\infty (x_n + y_n)^2$ is finite. Also, taking a square root, $(\sum_1^\infty (x_n + y_n)^2)^{1/2} \le (\sum_1^\infty (x_n)^2)^{1/2} + (\sum_1^\infty (y_n)^2)^{1/2}$.

On l_∞, if $|x_n| \le M$ and $|y_n| \le P$, then $|x_n + y_n| \le M + P$, which is an upper bound on an increasing sequence.

Finiteness of the three metrics is now easy. If $\{x_n\}$ and $\{y_n\}$ are in one of our spaces, so are $\{(-1)y_n\}$ and $\{x_n + (-1)y_n\}$ since we've shown any sum or scalar multiple of one of our sequences is also in the relevant space. The condition that $\{x_n - y_n\}$ is in the space implies that the relevant sum converges (for l_1 and l_2), or that $|x_n - y_n|$ is bounded and thus has a supremum (for l_∞).

The only thing left to check is that these alleged metrics really are metrics. Positivity and symmetry are trivial; I'll just check the triangle inequality. Given $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$, let $a_n = x_n - y_n$ and $b_n = y_n - z_n$. (Note that $\{a_n\}$ and $\{b_n\}$ are in our space, whichever space we are dealing with.) For l_1, we must show that $\sum_1^\infty |a_n + b_n| \le \sum_1^\infty |a_n| + \sum_1^\infty |b_n|$. But I already showed this, in the discussion of the sum of two sequences. For l_2, I already showed the analogous equation, $(\sum_1^\infty (a_n + b_n)^2)^{1/2} \le (\sum_1^\infty (a_n)^2)^{1/2} + (\sum_1^\infty (b_n)^2)^{1/2}$. For l_∞, $|a_n + b_n| \le |a_n| + |b_n| \le \sup |a_n| + \sup |b_n|$. Therefore, $\sup |a_n| + \sup |b_n|$ is an upper bound for $|a_n + b_n|$, so it must be greater than or equal to the least upper bound for $|a_n + b_n|$, i.e. $\sup |a_n + b_n| \le \sup |a_n| + \sup |b_n|$.

(b) We showed in Problem Set A that for $x, y \in \mathbf{R}^N$, we have $d_\infty(x, y) \le d_2(x, y) \le d_1(x, y)$. We can generalize this result to our spaces of sequences. For if $x = \{x_n\} \in l_1, y =$

$\{y_n\} \in l_1$, then $(\sum_1^N (x_n - y_n)^2)^{1/2} \le \sum_1^N |x_n - y_n| \le \sum_1^\infty |x_n - y_n| = d_1(\{x_n\}, \{y_n\})$. Therefore, $\sum_1^N (x_n - y_n)^2 \le (d_1(\{x_n\}, \{y_n\}))^2$, so letting N go to infinity (as I have done, more rigourously, several times already in this problem set) and taking the square root, we have $d_2(x, y) \le d_1(x, y)$ for $x, y \in l_1$.

Letting $\{x^{(i)}\} \subset l_1$, and plugging $y = x^{(i)}$ into the above inequality, we see that if $d_1(x^{(i)}, x) < \epsilon$ then $d_2(x^{(i)}, x) < \epsilon$. From this it follows from the definition of limit that if $\lim x^{(i)} = x$ with respect to d_1 then $\lim x^{(i)} = x$ with respect to d_2. (By the way, note that we need to use the fact that $l_1 \subset l_2$ to show that $d_2(x^{(i)}, x)$ is even defined.)

The same argument will show that $\lim x^{(i)} = x$ with respect to d_2 implies $\lim x^{(i)} = x$ with respect to d_∞, provided we can show $d_\infty(x, y) \le d_2(x, y)$ for $x, y \in l_2$. But for any N, $|x_N - y_N| \le max_{1 \le n \le N} |x_n - y_n| \le (\sum_1^N (x_n - y_n)^2)^{1/2}$ using that result from Problem Set A, so $|x_n - y_n| \le (\sum_1^\infty (x_n - y_n)^2)^{1/2} = d_2(x, y)$. Therefore, $d_2(x, y)$ is an upper bound for $x_n - y_n$, so it is greater than or equal to $\sup |x_n - y_n|$, so we have proved our relation between d_2 and d_∞. The argument I used for d_1 and d_2 works again, and completes the problem.

(c) $d_\infty(x^{(i)}, 0) = \sup\{1/i, 0\} = 1/i$, and $d_2(x^{(i)}, 0) = ((1/i)^2 + (1/i)^2 + \cdots (1/i)^2)^{1/2} = (i \cdot (1/i^2))^{1/2} = i^{-1/2}$. Both of these quantities can be made less than any given ϵ by taking i sufficiently large. That is, $x^{(i)} \to 0$ in d_2 and d_∞.

Let us suppose $x^{(i)} \to x$ in d_1, for some x, and try to derive a contradiction. By part (b), it's also true that $x^{(i)} \to x$ with respect to d_2, also. But $x^{(i)} \to 0$ in d_2, and a limit (if it exists) is unique, so we see that $x = 0$. However, $d_1(x^{(i)}, 0) = 1/i + 1/i + \cdots + 1/i = 1$. (Note that this is indeed finite, so $x^{(i)} \in l_1$, so it at least make sense to ask whether $x^{(i)} \to 0$.) This definitely can't be made smaller than arbitrary ϵ; for instance, if $\epsilon = 1/2$, then no choice of i will make $d_1(x^{(i)}, 0) < \epsilon$. So $x^{(i)} \not\to 0$, and we have a contradiction. This establishes that $x^{(i)}$ does not converge under d_1.

If we let $x_n^{(i)} = i^{-1/2}$ if $n \le i$, and $x_n^{(i)} = 0$ otherwise, it is easy to see that $d_\infty(x^{(i)}, 0) = i^{-1/2}$ and thus that $x^{(i)}$ in d_∞. However, $d_2(x^{(i)}, 0) = 1$. (Note again that this is finite, so $x^{(i)} \in l_2$.) The same argument as above shows that $x^{(i)}$ does not converge in d_2.

(d) Let $x = \{x_n\}_{n=1}^\infty$ and $y = \{y_n\}_{n=1}^\infty$. Define $d(x, y) = \sum_1^\infty n^{-2} \min\{1, |x_n - y_n|\}$. The idea of the "min" is that we chop off each distance if it gets too large, so we have something bounded to work with. That is, $\min\{1, |x_n - y_n|\} \le 1$, so the nth term in the infinite sum is less than or equal to $1/n^2$. Now, $\sum_1^\infty n^{-2}$ converges, a single-variable calculus fact I confess I've forgotten how to prove without the Integral Test. (Rudin proves this fact in Theorem 3.28, so you can look there for a proof.) It follows that any partial sum of our big ugly sequence is less than or equal to the corresponding partial sum of n^{-2}, which is less than or equal to the infinite sum of n^{-2}, so $\sum_1^\infty n^{-2} \min\{1, |x_n - y_n|\}$ converges. (This argument, which I have now used many times on this problem without stating a general theorem, is the Comparison Test. Look at Rudin Theorem 3.25 for a statement of the test.)

So, at least we have a well-defined function. Is it a metric? Symmetry and positivity are still easy (note that $\min\{1, a\}$ is 0 when $a = 0$ and is > 0 when $a > 0$). The triangle inequality is not too hard; just break it up into cases depending on whether various distances are more or less than 1, and use the triangle inequality for the standard metric on $\mathbf{R}$. (There may also be a slick way to prove the triangle inequality here, but I couldn't think of one.)

Finally, let's check that this metric has the desired property. Suppose $x_n^{(i)} \to x_n$ for all n. We want to show $x^{(i)} \to x$ under our metric on X. Let $\epsilon > 0$ be given. Find some M so that

$\sum_{M+1}^{\infty} n^{-2} < \epsilon/2$. We can do this because $\sum_1^{\infty} n^{-2}$ converges, so we can force the tail end to be as small as we like by starting the sum late enough (i.e., at M instead of at 1). (More rigourously you can use the fact that the partial sums form a Cauchy sequence to prove this.) Now that we've found M, let's look at $x_1^{(i)}, x_2^{(i)}, \ldots$, and $x_M^{(i)}$. Each of these converges to some limit; so for each $n \le M$, we can find an N_n such that $|x_n^{(i)} - x_n| < \epsilon/(2M)$ once $i > N_n$. Therrefore, $n^{-2}\min\{1, |x_n^{(i)} - x_n|\} \le \min\{1, |x_n^{(i)} - x_n|\} \le |x_n^{(i)} - x_n| < \epsilon/(2M)$. So, letting $N = \max\{N_1, \ldots, N_M\}$, we see that once $i > N \ge N_n$, we have $n^{-2}\min\{1, |x_n^{(i)} - x_n|\} < \epsilon/(2M)$; therefore $\sum_1^M n^{-2}\min\{1, |x_n^{(i)} - x_n|\} < \epsilon/2$. We have broken our original sum into two parts each of which is $< \epsilon/2$, so $d(x^{(i)}, x) < \epsilon$. (Yes, I'm add a finite sum to an infinite series and I haven't proved anything about that, but the gaps I'm leaving can be filled.)

So one direction of our claim is proved. We need only show that if $x^{(i)} \to x$ under our metric, then $x_n^{(i)} \to x_n$ for each n. Let n and $\epsilon > 0$ be given. We must show that $|x_n^{(i)} - x_n| < \epsilon$ for large enough i. But since $x^{(i)}$ converges, we can find N so large that when $i > N$, we have $d(x^{(i)}, x) < \epsilon/n^2$. So (looking back at our definition of d) we have an infinite series of nonnegative terms, whose sum is less than ϵ/n^2. Therefore, each term must be less than ϵ/n^2. In particular, the nth term must be less than that, which gives us $\min\{1, |x_n^{(i)} - x_n|\} < \epsilon$. Let's assume for now that $\epsilon < 1$ (after all, the important values of ϵ are the really small ones). Then $|x_n^{(i)} - x_n| < \epsilon$, once $i > N$., This takes care of $\epsilon < 1$. In particular, this works for $\epsilon = 1/2$. But that means we can make $|x_n^{(i)} - x_n| < 1/2 < \epsilon$ (by taking large enough i) if $\epsilon \ge 1$. So we're done for all values of ϵ. So $x_n^{(i)} \to x_n$ as desired. This completes the proof.

3. (a) We first show continuity of f_{ϵ,x_0} at an arbitrary point y_0 outside $B_\epsilon(x_0)$. In this case $f_{\epsilon,x_0}(y_0) = 0$. Let $\beta > 0$. We are looking for $\delta > 0$ such that whenever $y \in B_\delta(y_0)$, $f_{\epsilon,x_0}(y) \in B_\beta(f_{\epsilon,x_0}(y_0))$, or equivalently, $f_{\epsilon,x_0}(y) < \beta$ (as $f_{\epsilon,x_0}(y_0) = 0$ and the function takes on only nonnegative values).
Let $\delta = \epsilon\beta$. Let $y \in B_\delta(y_0)$. By the triangle inequality, $d(y, x_0) \ge d(x_0, y_0) - d(y, y_0) > \epsilon - \delta$. Then

$$f_{\epsilon,x_0}(y) = 1 - \frac{d(y, x_0)}{\epsilon} < 1 - \frac{\epsilon - \delta}{\epsilon} = \frac{\delta}{\epsilon} < \frac{\epsilon\beta}{\epsilon} = \beta, \tag{1}$$

as desired.
When y_0 is in $B_\epsilon(x_0)$, continuity of f_{ϵ,x_0} at y_0 is implied by continuity of the function $g : X \to \mathbf{R}$, $g(x) = d(x, x_0)$, which was proved in last week's problem set.

(b) Let us suppose that $\exists x \in X$ such that $\forall \epsilon > 0$, $B_\epsilon(x) \cap B_{\frac{1}{n}}(a_n) \neq \emptyset$ for infinitely many points a_n, making up a subsequence of a_n; let it be called b_n. For a certain ϵ and every n, let c_n be a point in $B_\epsilon(x) \cap B_{\frac{1}{n}}(b_n)$. By the triangle inequality, $d(x, b_n) \le d(x, c_n) + d(c_n, b_n) < \epsilon + \frac{1}{n}$. Hence for sufficiently large n and conveniently small ϵ we can get $d(x, b_n)$ to be as small as we want, i.e. b_n converges to x. This contradicts the hypothesis that a_n has no convergent subsequence, therefore $\forall x \in X, \exists \epsilon > 0$ such that $f_{\frac{1}{n},a_n}$ is zero on all points of the ball $B_\epsilon(x)$ (which by definition of f_{ϵ,x_0}, is equivalent to saying that $B_\epsilon(x) \cap B_{\frac{1}{n}}(a_n) = \emptyset$) for all but finitely many values of n.

(c)For every x, f is a finite sum of continuous functions on a neighborhood of x, hence f is continuous at x.

We want to show that f is unbounded, i.e. $\forall M > 0, \exists x \in X$ such that $f(x) > M$. Let $M > 0$ and choose $n_0 \in \mathbf{N}$ such that $n_0 > M$. $f(a_{n_0})$ consists of a finite number of positive terms one of which is $n_0 f_{\frac{1}{n_0}, a_{n_o}}(a_{n_0})$. Therefore $f(a_{n_0}) \geq n_0 f_{\frac{1}{n_0}, a_{n_0}}(a_{n_0}) = n_0 > M$, proving that f is unbounded. We have thus shown that if X is not compact, there exists a continuous unbounded function $f : X \to \mathbf{R}$. Equivalently, if all continuous functions $f : X \to \mathbf{R}$ are bounded, then X is compact.

4. Throughout this problem, recall that a map between metric spaces is continuous if and only if the inverse images of open sets are open, or equivalently, that the inverse images of closed sets are closed. Also understand that if X is a metric space and $Y \subset X$, then the open sets of the metric space Y (with the induced metric) are precisely the sets of the form $U \cap Y$, where U is an open subset of X (an easy exercise). The corresponding statement is true of closed sets.

(a) We are given a surjective continuous map $f : X \to \{0, 1\}$ where $\{0, 1\}$ has the discrete metric. Since the set $\{0\}$ is both open and closed, the preimage $f^{-1}(0)$ is both open and closed. Since f is surjective, this set is neither empty nor all of X, so we are done. Conversely, if there is a nonempty subset U of X not equal to X that is both open and closed, then define f to map elements of U to 0 and the elements of the complement U^c to 1. f is clearly continuous (just check inverse images of the only four possible open sets).

(b) X is connected and we have $f : X \to Y$ continuous. Observe that this defines a continuous map $f' : X \to f(X)$. (Why is f' continuous?) Now assume $f(X)$ is not connected. Then there is a continuous map from $f(X)$ to $\{0, 1\}$. And if we compose this with f', we obtain a continuous map from X to $\{0, 1\}$. This is a contradiction by part (a).

(c) Consider a connected subset S of $\mathbf{R}$. Now if $x, y \in S$ and a is between x and y, then a must be in S. (Otherwise, consider the set $[a, +\infty) \cap S = (a, +\infty) \cap S \subset S$.) Then let $a = \inf S$ and $b = \sup S$ with the obvious ∞ conventions. Since $a < x < b$ implies that $a < r < x < s < b$ for some $r, s \in S$ implies that $x \in S$, we obtain $(a, b) \subset S$. Finally, add a or b to (a, b) as appropriate.

(d) Consider an interval I in $\mathbf{R}$. Choose any nonempty $A \subset I$ that is both open and closed. And consider any $x \in A$. Now suppose that there exists $y \in I \backslash A$. Say that $x < y$ (the other case is the same argument). Let $z = \sup(A \cap [x, y])$. Since A is closed, $A \cap [x, y]$ is closed. Because z is a limit point of $A \cap [x, y]$, $z \in A \cap [x, y] \subset A$. In particular, $z < y$. Since A is open, we can find a ball around z contained in both A and in $[x, y]$, contradicting that z is the supremum.

(e) If $f : [a, b] \to R$ is continuous with $f(a) < 0$ and $f(b) > 0$, then $f([a, b])$ is connected by (d) and (b), thus an interval by (c). And 0 is in this interval since the interval contains a positive and negative value.

(f) Assume we have a homeomorphism $f : S^1 \to [0, 1]$. Choose x so that $f(x) \in (0, 1)$. Then the restriction map $f|_{S^1 \backslash \{x\}} : S^1 \backslash \{x\} \to [0, f(x)) \cup (f(x), 1]$ is a continuous map from a connected space *onto* a disconnected space, a contradiction by (b).

Two spaces X and Y are homeomorphic if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are the identity maps. (It is a simple exercise that this is equivalent to the existence of a continuous bijective f whose inverse is continuous.) So in our case, the map $f : (0, 1) \to S^1 \backslash \{1, 0\}$ defined as $f(t) = (\cos(2\pi t), \sin(2\pi t))$ is continuous. (It maps to $S^1 \backslash \{1, 0\}$ since $\cos^2 + \sin^2 = 1$. You can check also that $\{1, 0\}$ is not in the range.) As for a map g going in the other direction, define $g : S^1 \backslash \{1, 0\} \to (0, 1)$ as $g(x, y) = \frac{1}{2\pi} \arccos x$ when $y \geq 0$ and $1 - \frac{1}{2\pi} \arccos x$ when $y \leq 0$. By arccos we mean for the function to take its range in the interval $[0, \pi]$. This is continuous since it is continuous on the upper and lower halves and at the single point $\{-1, 0\}$. Lastly, verify that $f \circ g$ and $g \circ f$ are the identity.