Math 25a – Honors Advanced Calculus and Linear Algebra

Midterm Examination (November 12, 1997)

Name:
Harvard ID number:

Write your answers to each question on the same page as the question; if you need more space, continue your answer on the back. If you need to hand in extra sheets, be sure to **write your name on each sheet** you hand in.

1. Suppose V, W are finite-dimensional vector spaces, V' is a subspace of W, $T\colon V \to W$ and $S\colon W \to V$ are linear transformations, and $\{v_1, \ldots, v_n\}$ is a basis for V. Answer the following questions true or false; "true" if they are true for **all** choices of V, V', W, T, S and B, "false" otherwise.

   (a) $\{v_1, \ldots, v_n\} \cap V'$ is a basis for V'.

   (b) $\{2v_1, 2v_2, \ldots, 2v_n\}$ is a basis for V.

   (c) $\{T(v_1), \ldots, T(v_n)\}$ spans $\operatorname{Im} T$.

   (d) $\{T(v_1), \ldots, T(v_n)\}$ is a basis for $\operatorname{Im} T$.

   (e) if $S \circ T$ is injective, then S is injective.

   (f) if $S \circ T$ is injective, then T is injective.

   (recall that we defined $S \circ T(v) = S(T(v))$.)

   Answers: (a) (b) (c) (d) (e) (f)

2. Let (X, d_X), (Y, d_Y) be metric spaces.

- Write carefully the definition of when a subset $A \subset X$ is open.

- Write a statement expressing that a function $f\colon X \to Y$ is **not** continuous, without using the word "not" or negative expressions like "for no a", etc.

- Prove that f is continuous if and only if for all open sets $U \subset Y$, the set

  $$f^{-1}(U) = \{x \in X \mid f(x) \in U\}$$

  is open.

3. Suppose $L_1, L_2 \colon V \to \mathbb{R}$ are linear transformations from a vector space V to $\mathbb{R}$, and that the nullspaces (kernels) of L_1 and L_2 are the same. Prove that L_1 is a nonzero multiple of L_2, i.e. there is a nonzero real number a with $L_1 = aL_2$.

4. $\mathcal{P}_3$ is the vector space of polynomials of degree at most 3. Define a linear transformation $T\colon \mathcal{P}_3 \to \mathbb{R}^3$ by $T(f) = (f(2), f'(1), 3f(0) + f'''(0))$. Compute

   (a) the matrix of T using the standard basis of $\mathbb{R}^3$ and the basis $\{1, x, x^2, x^3\}$ of $\mathcal{P}_3$,

   (b) the rank of T, and

   (c) a basis for the nullspace $\mathcal{N}(T)$.

   (parts (b) and (c) require proofs.)

5. Define a function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x_1, x_2) = (x_1^2 + x_2^2, x_1 x_2)$. Calculate the Jacobian matrix $J_f(x_1, x_2)$, and describe, with proof the sets of points $\vec{v} = (x_1, x_2)$ where

   (a) $D_{\vec{v}} f$ has rank 0.

   (b) $D_{\vec{v}} f$ has rank 1.

6. Suppose a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\vec{0}$ and that $\vec{v}$ is in the nullspace of $D_{\vec{0}}f$. Define a function $g\colon \mathbb{R} \to \mathbb{R}$ by $g(t) = f(t\vec{v})$. Prove that g is differentiable at 0, and that $g'(0) = 0$.