

Math 25b – Problem Set 1, due Friday, February 6.

1. CS, p. 282 #3, #4 bd, #5 bd, #8 (Note: in #8, assume $f \in \mathcal{C}^1$)
2. CS, p. 296 #5
3. CS, p. 301 #1 bdf
4. In this problem you will investigate the “scalar curvature” of a \mathcal{C}^2 path $\alpha: U \rightarrow \mathbb{R}^n$, where U is an open set in \mathbb{R} .

(a) We say the path α is “parametrized by arc length” if $\|\alpha'(t)\| = 1$ for all $t \in U$. (Note: all norms in this problem come from the standard inner product on \mathbb{R}^n). Suppose that $\alpha'(t) \neq 0$ for all t . Assuming the ordinary one-variable fundamental theorem of calculus, show that there is an open set $U' \subset \mathbb{R}$ and a surjective function $s: U' \rightarrow U$ so that the reparametrized path $\tilde{\alpha} = \alpha \circ s$ is parametrized by arc length. (Let $r(t) = \int_0^t \|\alpha'(x)\| dx$ and apply the (strong) inverse function theorem to r).

For the rest of the problem, assume α is parametrized by arc length.

- (b) Show that the vectors $\alpha'(t)$ and $\alpha''(t)$ are perpendicular for all t . (Hint: differentiate $\|\alpha'(t)\|^2$)
- (c) We define the scalar curvature of α at time t to be the number $\kappa(t) = \|\alpha''(t)\|$. Calculate κ if $\alpha(t) = (\cos t, \sin t, t)$ traces a helix in \mathbb{R}^3 . Note that α is not parametrized by arc length.
- (d) Suppose we know that $\kappa(0) \neq 0$. Take a real number r and consider the point $\vec{c} = \alpha(0) + r\alpha''(0)$. Define a function f by $f(t) = \|\alpha(t) - \vec{c}\|^2$, and show that $f'(t) = 2\langle \alpha(t) - \vec{c}, \alpha'(t) \rangle$. Deduce that $f'(0) = 0$. Show that $f''(0) = 0$ if and only if $r = 1/(\kappa(0))^2$.

The circle with center \vec{c} and radius $1/\kappa(0)$, lying in the plane spanned by $\alpha'(0)$ and $\alpha''(0)$ is called the “osculating circle” to the curve at $\alpha(0)$.

5. We showed in lecture how to use the implicit function theorem to pass from an implicit description of a manifold (as a level set of a function $\mathbb{R}^{n+k} \rightarrow \mathbb{R}^k$) to an explicit parametrization. Now we will go in the other direction.

Let $f: U \rightarrow \mathbb{R}^{n+k}$ be a \mathcal{C}^1 function on an open set $U \subset \mathbb{R}^n$, and assume its derivative has rank n at $\vec{v}_0 \in U$, i.e. assume the derivative is injective. Consider f as consisting of two functions: $f_1: U \rightarrow \mathbb{R}^n$ given by the first n coordinate functions of f and $f_2: U \rightarrow \mathbb{R}^k$ given by the last k . Without loss of generality assume that $D_{\vec{v}_0} f_1$ is invertible.

Apply the inverse function theorem to f_1 to obtain an inverse function f_1^{-1} defined on an open set U' containing $f_1(\vec{v}_0)$. Use this show that there is an open set V in \mathbb{R}^{n+k} containing $\vec{w}_0 = f(\vec{v}_0)$ and a \mathcal{C}^1 function $g: V \rightarrow \mathbb{R}^k$ so that $g \circ f = 0$, g vanishes only on points of $(\text{Im } f) \cap V$, and the derivative $D_{\vec{w}_0} g$ has rank k . (Hint: the image of the function $f \circ f_1^{-1}$ looks like a graph)