Math 25b – Solution Set 1
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1. #3 f is clearly $\mathcal{C}^1$.

$$Df = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$\det Df = r\cos^2\theta + r\sin^2\theta = r\left(\cos^2\theta + \sin^2\theta\right) = r$ So $\det Df = 0 \iff r = 0$, so at every point on its domain, Df is an isomorphism, so by the Inverse Function Theorem, f is locally invertible. $f(1,0) = f(1,2\pi) = (1,0)$, so f is not injective, so not globally invertible.

Remember, when you claim that a function is not invertible (or similar claims), all that you need to do is give one counterexample – but you must give a counterexample. For instance, writing "$f(r,\theta) = f(r,\theta+2\pi)$, so f is not injective" is not satisfactory, because you also need to show that there are points $(r,\theta), (r,\theta+2\pi)$ in your domain that satisfy the equation.

#4 Note that by the Strong Inverse Function Theorem, it is necessary and sufficient that $f'(x) \neq 0$ on our interval.

(b) $f'(x) = 3x^2 - 10x + 3 = (3x-1)(x-3) \implies (f'(x) \neq 0 \iff x \neq 1/3, 3)$. $x_0 = 0$, so our interval is $(-\infty, 1/3)$.

(d) $f'(x) = \cos x \implies (f'(x) \neq 0 \iff x \neq (k+1/2)\pi, k$ an integer$)$. $x_0 = -1$, so our interval is $(-\pi/2, \pi/2)$.

#5 By Inverse Function Theorem, $(f^{-1})'(y_0)) = (f(x_0))^{-1}$

(b) $1/3$

(d) $1/\cos(-1) = \sec 1$

#8 (a) Given balls $B_r(x_0), B_{r'}(x_1)$, translate the first ball to the origin, dilate (stretch or shrink) to proper radius, and translate to new center. So $f = U \circ T \circ S$, where $S(x) = x - x_0$, $T(x) = \frac{r'}{r}x$, $U(x) = x + x_1$. Each of these is clearly differentiable, and we can analogously invert f to show that it is a diffeomorphism.

(b) No. Consider f from 1, with $U = U' = \mathbb{R}^2 \setminus \{(0,0)\}$. f is not injective, so it isn't even a homeomorphism.

(c) I assume that Corwin and Szczarba intended $f \in \mathcal{C}^1$, otherwise we don't have the tools to prove this, and it may be false. If we assume this, then by bijection $\exists f^{-1}$, and this inverse is differentiable because at each point it is differentiable, by Inverse Function Theorem. Note that differentiability is a local property, so we only need to have it at each point. This is important, as it enables us to generalize differentiation to spaces that only are locally $\mathbb{R}^n$, namely manifolds.

2. #5 Let us denote the set of points on the lemniscate by L, and $X = \left\{(x,y)|\frac{\partial f}{\partial x} = 0\right\}$, $Y = \left\{(x,y)|\frac{\partial f}{\partial y} = 0\right\}$.

$$\frac{\partial f}{\partial x} = 2(x^2+y^2)\cdot 2x - 2a^2\cdot 2x = 4x(x^2+y^2-a^2)$$

$$\frac{\partial f}{\partial y} = 2(x^2+y^2)\cdot 2y + 2a^2\cdot 2y = 4y(x^2+y^2+a^2)$$

Then $X = \{(x,y)|(x=0) \text{ or } (x^2+y^2=a^2)\}, Y = \{(x,y|y=0)\}$.

(a) The answer is $L\cap X\cap Y$; the points on the lemniscate where both partials vanish. $X \cap Y = \{(x,y)|(y=0) \text{ and } ((x=0) \text{ or } (x=\pm a))\} = \{(0,0),(\pm a,0)\}$. To find $Y \cap L$, set $y=0$ and solve the lemniscate's equation for x:

$$0 = (x^2)^2 - 2a^2x^2$$
$$0 = x^2(x^2-2a^2)$$
$$x = 0, \pm a\sqrt{2}$$

So $Y \cap L = \left\{(0,0),(\pm a\sqrt{2},0)\right\}$. So $L\cap X\cap Y = \{(0,0)\}$.

(b) This is just $Y \cap L$, above.

$$\frac{dy}{dx} = -\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial y}\right)^{-1} = \frac{-x(x^2+y^2-a^2)}{y(x^2+y^2+a^2)}$$

3. #1 For $f\colon \mathbb{R}^n \to \mathbb{R}$, $J_f = \left[\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}\cdots\frac{\partial f}{\partial x_n}\right]$ so if we define, say, x_n implicitly (denote by g) we obtain $J_g = -B^{-1}A = -\frac{\partial f}{\partial x_n}^{-1}\left[\frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2}\cdots\frac{\partial f}{\partial x_{n-1}}\right]$ so $\frac{\partial x_n}{\partial x_i} = \frac{-\partial F/\partial x_i}{\partial F/\partial x_n}$ as we would expect from the case $n=2$. In particular, $\partial z/\partial x = \frac{-\partial F/\partial x}{\partial F/\partial z}$; $\partial z/\partial y = \frac{-\partial F/\partial y}{\partial F/\partial z}$.

(b) $\partial F/\partial z = x+y-xy; \partial F/\partial z(2,2,-3) = 2+2-2\cdot 2 = 0$
Nope – doesn't define it implicitly.

(d) $\partial F/\partial z = 1-3yz^2; \partial F/\partial z(1,2,3) = 1-54 = -53$
Yup – defines it implicitly.
$\partial F/\partial x = y^6 - y; \partial F/\partial x(1,2,3) = 64-2 = 62$ $\partial F/\partial y = 6xy^5 - z^3 - x; \partial F/\partial y(1,2,3) = 192-27-1 = 164$
Thus, $\partial z/\partial x(1,2,3) = 62/53; \partial z/\partial y(1,2,3) = 164/53$.

(f) $\partial F/\partial z = 1-\sin xyz; \partial F/\partial z(0,0,0) = 1$
Yup – defines it implicitly.
$\partial F/\partial x = \partial F/\partial y = \partial F/\partial z = 1-\sin xyz$; $\partial F/\partial x(0,0,0) = \partial F/\partial y(0,0,0) = \partial F/\partial z(0,0,0) = 1$.
Thus, $\partial z/\partial x(0,0,0) = \partial z/\partial y(0,0,0) = -1/1 = -1$.

4. (a) Let $r(t) = \int_0^t \|\alpha'(x)\|\, dx$. By the Fundamental Theorem of Calculus, $r'(t) = \|\alpha'(t)\| > 0$, so we apply the Strong Inverse Function Theorem and obtain $s = r^{-1}$. Let $U' = r(U)$. Then $s\colon U' \to U$ bijectively, $s'(r(t)) = 1/r'(t) = 1/\alpha'(t)$ so in particular $s \in \mathcal{C}^2$. Further, $s'(t) = s'(r(s(t))) = 1/r'(s(t)) = 1/\alpha'(s(t))$. So $\tilde{\alpha}(t) = \alpha(t) \circ s, \tilde{\alpha}'(t) = \alpha'(s(t)) \cdot s'(t) = \alpha'(s(t))/\alpha'(s(t)) = 1$.

   (b) $1 = \|\alpha'(t)\| = \langle \alpha'(t), \alpha'(t) \rangle$. Recall that $\langle f, g \rangle' = \langle f', g \rangle + \langle f, g' \rangle$ (this can be determined by differentiating the inner product sum term-wise). So $0 = 1' = \langle \alpha'(t), \alpha'(t) \rangle' = \langle \alpha''(t), \alpha'(t) \rangle + \langle \alpha'(t), \alpha''(t) \rangle = 2\langle \alpha''(t), \alpha'(t) \rangle$, so $\alpha'(t) \perp \alpha''(t)$.

   (c) This curve is not parametrized by arc length. So apply part 4a (it's not that bad!): $\|\alpha'(t)\| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2}$. So $r(t) = \int_0^t \sqrt{2} dx = \sqrt{2}t$, so $s(t) = r^{-1}(t) = t/\sqrt{2}$. So $\tilde{\alpha} = \alpha \circ s = (\cos t/\sqrt{2}, \sin t/\sqrt{2}, t/\sqrt{2})$. So $\tilde{\alpha}'' = (1/2) \cdot (\cos t/\sqrt{2}, \sin t/\sqrt{2}, 0), \kappa(t) = (1/2)\sqrt{\cos^2 t/\sqrt{2} + \sin^2 t/\sqrt{2}} = 1/2$.

   (d) $f'(t) = \langle \alpha(t) - \vec{c}, \alpha(t) - \vec{c} \rangle' = \langle \alpha'(t) - (\vec{c})', \alpha(t) - \vec{c} \rangle + \langle \alpha(t) - \vec{c}, \alpha'(t) - (\vec{c})' \rangle = 2\langle \alpha(t) - \vec{c}, \alpha'(t) \rangle$.
   $f'(0) = 2\langle \alpha(0) - (\alpha(0) + r\alpha''(0)), \alpha'(0) \rangle = -2r\langle \alpha''(0), \alpha'(0) \rangle = 0$ (by part 4b).
   $f''(t) = 2\langle \alpha(t) - \vec{c}, \alpha'(t) \rangle' = 2\langle \alpha'(t), \alpha'(t) \rangle + 2\langle \alpha(t) - \vec{c}, \alpha''(t) \rangle = 2 + 2\langle \alpha(t) - \vec{c}, \alpha''(t) \rangle$.
   So $f''(0) = 2 + 2\langle \alpha(0) - (\alpha(0) + r\alpha''(0)), \alpha''(0) \rangle = 2 - 2r\langle \alpha''(0)), \alpha(0)'' \rangle = 2(1 - r\|\alpha''(t)\|^2) = 2(1 - r(\kappa(0))^2) = 0 \iff 1 = r(\kappa(0))^2 \iff r = 1/(\kappa(0))^2$.

5. The notation for this problem is the greatest difficulty. Apply the Inverse Function Theorem to f_1 to get $U^* \subset U, f_1\colon U^* \to U'$ a diffeomorphism with $f_1(U^*) = U'$. So we have an inverse function $f_1^{-1}\colon U' \subset \mathbb{R}^n \to U^* \subset U \subset \mathbb{R}^n$. We want functions from $\mathbb{R}^{n+k} \to \mathbb{R}^k$. But we don't have any functions from $\mathbb{R}^{n+k}$. Or do we? Whenever you have a product $X \times Y$, you have projections $\pi_X\colon X \times Y \to X$, $\pi_Y\colon X \times Y \to Y$, with $\pi_X(x, y) = x$, $\pi_Y(x, y) = y$ (note that we use π for a projection). You've probably used these before without giving them a second thought. So we have maps $\pi_n\colon \mathbb{R}^{n+k} \to \mathbb{R}^n$, $\pi_k\colon \mathbb{R}^{n+k} \to \mathbb{R}^k$. The only map that we have from $\mathbb{R}^n \to \mathbb{R}^k$ is f_2 (well, we could project again, but we want to use f_2). Lastly, we want to use f_1^{-1} at some point, so we should use it the only place we can: after projecting, before applying f_2. So we now have two maps $\mathbb{R}^{n+k} \to \mathbb{R}^k$, namely $f_2 \circ f_1^{-1} \circ \pi_n$, and π_k. Since these are vector spaces, we may wish to add or subtract these maps – and since we want to get zero (i. e., cancellation), let's subtract them. So our map should be $\pi_k - (f_2 \circ f_1^{-1} \circ \pi_n)$. The point of this is that you can often figure out a confusing problem by just looking at what tools you have, and applying them.

   Okay, so take $V = U' \times \mathbb{R}^k$ (note that we can't just take $f(U^*)$, as this may not be open – consider the case where f_2 is the zero map). This clearly contains $\vec{w}_0 = f(\vec{v}_0)$. Rewriting $\pi_k - (f_2 \circ f_1^{-1} \circ \pi_n)$, consider $w = (w_n, w_k), w \in \mathbb{R}^{n+k}, w_n \in \mathbb{R}^n, w_k \in \mathbb{R}^k$; then our function is $g(w_n, w_k) = w_k - f_2(f_1^{-1}(w_n))$. Intuitively, what is going on? We're taking a point in R^{n+k}, looking at its first n coordinates, seeing what the next k coordinates should be $(f_2 \circ f_1^{-1})$, and then subtracting. So the result will be zero iff the last k coordinates are in order, namely iff we are on the manifold. So it vanishes only on $(\operatorname{Im} f) \cap V$, as desired. Plugging in f, we find that $g \circ f(v) = f_2(v) - f_2(f_1^{-1}(f_1(w_n))) = f_2(v) - f_2(v) = 0$. Projections are continuous, differentiable, and everything in the world that you could possibly want *except* bijective. Oh, and they don't give massages. Sorry. Anyway, so $g \in \mathcal{C}^1$, as it is the composition of $\mathcal{C}^1$ functions. Lastly, $J_g(w) = [*|I_k]$, where $*$ is something that depends on f_1, f_2. But we don't care, since the last k coordinates just get projected. So $\text{rank}(J_g(w)) = k$.