Math 25b – Solution Set 4
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1. 2 This follows by writing out each side. Note that this means that complex conjugation is a field automorphism of the complex numbers (that means that it respects addition and multiplication, and is invertible—automorphism are analogous to invertible linear transforms, only on fields instead of vector spaces).

   4 Please excuse Corwin and Szczarba. Their definition of polar coordinates is incorrect: it yields the same polar coordinates for $(1,1),(-1,-1)$. Indeed, it gives the incorrect angle for any complex number with non-positive real part (and isn't even defined for purely imaginary numbers). Kudos to everyone who noticed that. Shame on us for not noticing this and telling you.

   (a) $\left|e^{i\theta}\right|^2 = e^{i\theta}\overline{e^{i\theta}} = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = (\cos\theta)^2 + (\sin\theta)^2 = 1$. Since $|\ | \geq 0$, $\left|e^{i\theta}\right| = 1$.

   (b) Using the *correct* definition of polar coordinates, we see that the resulting r, θ work.

   (c) Note that $|zw| = |z|\,|w|$; this follows from commutativity of complex multiplication and automorphism of complex conjugation. Thus, given $(r,\theta),(r',\theta')$ that give z, $\left|re^{i\theta}\right| = |r|\left|e^{i\theta}\right| = |r|$ and similarly for r'. So $|r| = |r'|$, and since $r, r' \geq 0$, $r = r'$.

   (d) This follows by writing them out, and using trig identities. Note that the second follows by induction by putting in $r_1 = r, r_2 = r^n, \theta_1 = \theta, \theta_2 = n\theta$. It is also possible to first define exp, show that $e^x e^y = e^{x+y}$, and from this easily derive all the trig identities.

   5 $f'(t) = (\cos t + i\sin t)' = -\sin t + i\cos t = i(\cos t + i\sin t) = if(t)$.

2. 4 If T is unitary, then it has an orthonormal eigenbasis, and is a fortiori diagonalizable[1]. In such a basis, $T = d(\lambda_1,\ldots,\lambda_n)$, and thus $T^* = \overline{T^t} = d(\overline{\lambda_1},\ldots,\overline{\lambda_n})$. Now by unitary, $T^*T = I$, so $\lambda_i\overline{\lambda_i} = 1$, so $|\lambda_i|^2 = 1$, and since $|\ | \geq 0$, $|\lambda_i| = 1$.

   6 $(-T)(v) = cv \iff T(-v) = cv = -c(-v)$, so the eigenvalues of $-T$ are exactly the additive inverses of the eigenvalues of T.

   7 If $T(v) = cv$, then $(T^2)(v) = T(T(v)) = T(cv) = (cT)(v) = c(cv) = c^2v$, so for every eigenvalue of T, we get its square as an eigenvalue for T^2. It is not clear that these are the only eigenvalues; consider rotation by $\pi/2$—this is still true, but you need complex numbers. This will definitely be true if T is diagonalizable; and also in general, though it takes a little doing. Suppose c is an eigenvalue for T^2, with eigenvector v. Then we claim $\sqrt{c}$or$-\sqrt{c}$ is an eigenvalue for T. Consider $w = \sqrt{c}v + Tv$. This is nonzero if $Tv \neq -\sqrt{c}v$ (in case of equality, we are done). But $Tw = T(\sqrt{c}v + Tv) = \sqrt{c}Tv + T^2v = \sqrt{c}Tv + cv = \sqrt{c}w$, so $\sqrt{c}$ is an eigenvalue for T.

[1] You don't need the spectral theorem, but it makes for a brief proof.

8 By arguments similar to the above, the eigenvalues of kT are $k\lambda_i$, those of T^n are λ_i^n, and those of $A+B$ are a_i+b_i (assuming that A, B have the same eigenvectors, with eigenvalues a_i, b_i). So by induction, the eigenvalues of $P(T)$ contain $P(\lambda_i)$.

To show that these are the only ones, it is easiest to use Jordan normal form (which says that every matrix is almost diagonalizable). You might also be able to do this directly by an induction, but this is far too painful. We know, due to the above, that the eigenvalues of kT are exactly $k\lambda_i$, and that the eigenvalues of T^n are exactly λ_i^n, but finding eigenvalues/vectors for $A+B$ in terms of those of A, B is generally very difficult—consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

3. For $1, 5$ I have just listed the solutions, as writing out lots of calculations is not very instructive, and a real pain in LaTeX. Note that to find eigenvalues, you just write out the characteristic polynomial and factor, and to get the eigenvectors you don't just guess and write down the answer—you find the kernel of $\lambda I - T$ and choose some vectors that span it. Remember to normalize!

1 (d) $1/\sqrt{2}(1,1), 1/\sqrt{2}(1,-1)$,

(f) $1/\sqrt{6}(1-2i,-1), 1/\sqrt{6}(1,1+2i)$, Your answer may look different—just multiply by complex numbers and everything works out.

5 $TT^* = T^*T = d(17,17)$. Basis: $1/\sqrt{2}(1,i), 1/\sqrt{2}(1,-i)$.

8 (a) In this orthonormal basis, both T, T^* are diagonal, and diagonal matrices commute, so $TT^* = T^*T$.

Why do you need to assume normal? That is, why doesn't diagonalizable imply normal? Because T, T^* may not be simultaneously diagonalizable, i. e., there may not be a basis in which they are both diagonal.

(b) If all the eigenvalues are real, then in this basis, T is diagonal with real entries, so $T^* = \overline{T^t} = T$, as desired.

4. (a) We show the equivalence of the first three, and of the last two. We then show the equivalence of $(1),(4)$, so they are all equivalent.

(1) $\iff$ (2) Over $\mathbb{R}, \overline{A} = A$, so $A^* = A^t$. So $A^* = A^{-1} \iff A^t = A^{-1}$.

(2) $\iff$ (3) If we denote the columns of A by $v_1, \ldots, v_n$, then $(A^tA)_{ij} = v_i \cdot v_j$, so $A^tA = I$ iff $v_1, \ldots, v_n$ are orthonormal. Orthogonal vectors are linearly independent, and orthonormal vectors are non-trivial, so a maximal set of vectors form an orthonormal basis iff they are orthonormal.

(4) $\implies$ (5) If $\forall \vec{v}, \vec{w} : \langle \vec{v}, \vec{w} \rangle = \langle T\vec{v}, T\vec{w} \rangle$, then in particular $\|v\|^2 = \langle \vec{v}, \vec{v} \rangle = \langle T\vec{v}, T\vec{v} \rangle = \|Tv\|^2$, and since $\|v\|, \|Tv\| \geq 0$, $\|v\| = \|Tv\|$.

(5) $\implies$ (4) If $\forall \vec{v}, \|v\| = \|Tv\|$, then (by polarization identity, and linearity of

T):

$$\begin{aligned}\langle T\vec{v}, T\vec{w}\rangle &= \frac{1}{2}\left(\|Tv+Tw\|^2 - \|Tv-Tw\|^2\right)\\ &= \frac{1}{2}\left(\|T(v+w)\|^2 - \|T(v-w)\|^2\right)\\ &= \frac{1}{2}\left(\|v+w\|^2 - \|v-w\|^2\right)\\ &= \langle \vec{v}, \vec{w}\rangle\end{aligned}$$

This and the above show that an inner product is equivalent to a norm, and that a linear transform preserves one iff it preserves the other.

(1) $\iff$ (4) By definition of adjoint, $\langle T\vec{v}, T\vec{w}\rangle = \langle \vec{v}, T^*T\vec{w}\rangle$. We claim that $\langle \vec{v}, \vec{w}\rangle = \langle \vec{v}, T^*T\vec{w}\rangle \forall \vec{v}, \vec{w} \iff T^*T = I$; from this (1) $\iff$ (4) follows. $\Leftarrow$ is clear; what is not obvious is $\implies$. However, note that $\langle \vec{v}, \vec{w}\rangle = \langle \vec{v}, B\vec{w}\rangle \forall \vec{v}, \vec{w} \implies B = I$, because $\langle \vec{v}, \vec{w}\rangle = \langle I\vec{v}, \vec{w}\rangle$, so *by uniqueness of adjoint*, $B = I^* = I$.

(b) The columns of an invertible matrix A form a basis, so applying Gram-Schmidt, we obtain $AU' = O$, where U' is upper triangular (since you only need to use the first k vectors to get the k-th orthonormal vector), and the columns of O are orthonormal, by construction, so O is orthogonal, by property (3). The inverse of an upper triangular matrix is upper triangular, so set $U = U'^{-1}$, and we get $A = OU$, as desired. Note that by taking transposes, we get $B = LO'$, where L is lower triangular, O' is orthogonal.

5. (a) If we have V, a vector space, with $\dim V = n$, then $\mathcal{L}(V, V)$ is a vector space, with $\dim \mathcal{L}(V, V) = n^2$. Consider $\left\{T^0, T^1, \ldots, T^{n^2}\right\}$. This is a set of $n^2 + 1$ vectors in $\mathcal{L}(V, V)$, so there is a nontrivial linear dependence: $a_0T^0 + a_1T^1 + \cdots + a_{n^2}T^{n^2} = 0$. This is our polynomial.

(b) Consider $S = \{\deg P_T | P_T(T) = 0, P_T \neq 0\}$ (the set of the degrees of all non-trivial polynomials that annihilate T). This is a set of natural numbers, and is non-empty by 5a. Every non-empty set of natural numbers has a least element, by well-ordering of the natural numbers[2]. Therefore there is a minimal degree.

To show that there is a unique monic polynomial of minimal degree (a monic polynomial is one whose leading coefficient is 1), suppose that we have two minimal polynomials, P_T, P'_T. Then $(P_T - P'_T)(T) = P_T(T) - P'_T(T) = 0 - 0 = 0$, and $\deg(P_T - P'_T) < \deg P_T = \deg P'_T$, since the leading terms cancel. So by minimality of $\deg P_T$, $P_T - P'_T = 0$, so $P_T = P'_T$, as desired. You could also have done this by contradiction, but that is unnecessary.

(c) Factor $P_T(x) = (x - \lambda_1)\ldots(x - \lambda_d)$. Note that order doesn't matter, as T, I commute with each other and themselves. Also, note that if A is invertible, $\ker AB = \ker B$. So if $T - \lambda_i I$ is invertible, then factoring it out will not change

[2] An ordered set is well-ordered iff every non-empty subset contains a least element; the natural numbers are well ordered, but the integers, (positive) rationals and (positive) reals are not.

the kernel of $P_T(T)$. But this contradicts minimality, so $T - \lambda_i I$ is not invertible, so λ_i is a characteristic value of T. So in our case, $\lambda_i = 0$, so $P_T(x) = x^d$, so $P_T(T) = T^d = 0$.

It is not immediately obvious, but the characteristic polynomial of T annihilates T. Further, the minimal polynomial of T divides the characteristic polynomial. We showed above that every root of the minimal polynomial is a root of the characteristic polynomial—it is not hard to show that every root of the characteristic polynomial is also a root of the minimal polynomial: for every eigenvalue c, take an eigenvector α and apply $P_T(T)$ to it; then by 2, $P_T(T)\alpha = P_T(c)\alpha$—but this equals zero, since P_T annihilates T, so $P_T(c) = 0$, so c is a root of P_T. From this it follows that they have the same roots, with possibly different multiplicities. What is a little hard to show is that the characteristic polynomial annihilates T. Yup, lots of interesting results in linear algebra.