

Math 25b – Problem Set 5, due Friday, March 13.

Consider a surface $S = f^{-1}(0)$, where $f: U \rightarrow \mathbb{R}$ is a \mathcal{C}^2 function defined on an open set $U \subset \mathbb{R}^3$. Suppose that $\text{grad } f \neq \vec{0}$ at all points of U . Define a function $\vec{n}: U \rightarrow \mathbb{R}^3$ by $\vec{n}(p) = \text{grad } f(p) / \|\text{grad } f(p)\|$. Then at points $p \in S$, the vector $\vec{n}(p)$ is a unit vector perpendicular to the tangent space $T_p S$.

Take a \mathcal{C}^2 path $\alpha: V \rightarrow S$, where $V \subset \mathbb{R}$ is an open set containing 0. Assume α is parametrized by arc length. Put $p = \alpha(0)$, $\vec{v} = \alpha'(0)$, $\vec{n} = \vec{n}(p)$.

1. Show that

$$\alpha''(0) = \kappa_s \vec{n} + \vec{w}$$

for some number $\kappa_s \in \mathbb{R}$ and some vector \vec{w} which is perpendicular to both \vec{v} and \vec{n} . Deduce that $\kappa_s = \langle \alpha''(0), \vec{n} \rangle$. κ_s is called the *sectional curvature* of the path, and $\kappa_g = \|\vec{w}\|$ is called the *geodesic curvature*. Note that the curvature of α is given by $\kappa = \sqrt{\kappa_s^2 + \kappa_g^2}$.

2. Show that a different choice of equation defining S will at most multiply \vec{n} by -1 , so the absolute value $|\kappa_s|$ is defined independent of f .
3. Calculate κ_s and κ_g at time $t = 0$ for the two paths $\alpha_1 = (5 \cos t, 5 \sin t, 0)$ and $\alpha_2 = (3 \cos t, 3 \sin t, 4)$ contained in the surface $S = \{(x, y, z) \mid x^2 + y^2 + z^2 = 25\}$. Remember to reparametrize!
4. Differentiate $\langle \alpha'(t), \vec{n}(\alpha(t)) \rangle$ and show that $\kappa_s = -\langle \vec{v}, D_p \vec{n}(\vec{v}) \rangle$. Thus the sectional curvature of the path only depends on the velocity vector \vec{v} . The quadratic form $Q(\vec{v}) = -\langle \vec{v}, D_p \vec{n}(\vec{v}) \rangle$ defined on $T_p S$ is called the *second fundamental form* of S at p .
5. Show that there is an orthonormal basis $\{\vec{v}_1, \vec{v}_2\}$ of $T_p S$ and numbers κ_1, κ_2 so that $Q(a\vec{v}_1 + b\vec{v}_2) = a^2 \kappa_1 + b^2 \kappa_2$.

The vectors \vec{v}_1 and \vec{v}_2 are well-defined up to a sign, and are called the *principal directions* of S at p . The numbers κ_1 and κ_2 are the *principal curvatures*. Their product $G = \kappa_1 \kappa_2$ is called the *Gaussian curvature*; the sum $H = \kappa_1 + \kappa_2$ is called the *mean curvature*. Notice that G is completely independent of f !

6. Suppose S is the graph of a \mathcal{C}^2 function $F(x, y)$; take $f(x, y, z) = z - F(x, y)$ as an equation for S . Suppose we have $\text{grad } F(x_0, y_0) = \vec{0}$ and let $p = (x_0, y_0, F(x_0, y_0))$. Then $T_p S$ consists of vectors of the form $\vec{v} = (a, b, 0)$.

Show that the partial derivatives of $\|\text{grad } f\|$ vanish at p . Use this to show that the second fundamental form of S is given by $Q((a, b, 0)) = \langle (a, b), H_F(p) \cdot (a, b) \rangle$, where H_F is the Hessian matrix.

Explain how the numbers G and H tell you whether F has a local maximum, minimum, or saddle at (x_0, y_0) .

7. (still optional!) Now we can answer the challenge problem from PS#2. Let f and F be as in the last part, but assume that $\|\text{grad } F(x, y)\| = 1$, and $F(x, y) = 0$. Therefore the set $C = f^{-1}(0)$ is a smooth curve near (x, y) . Think of C as a curve in the xy plane in \mathbb{R}^3 , so $C \subset S$.

Consider the following two paths α and β in S , both passing through p at time $t = 0$. Let α trace out the curve C , parametrized by arc length, and so that

$$\alpha'(0) = \vec{v} = (\partial F / \partial y, -\partial F / \partial x, 0)|_p.$$

Let L be the straight line through p in the direction of \vec{v} , and let β trace the curve in S which lies over L .

Show that α and β have the same curvature (hint: $\alpha''(0)$ and $\beta''(0)$ both make a 45° angle with the normal vector \vec{n}).

Show that the curvature of β is given by

$$\kappa = |\langle \vec{w}, H_F \cdot \vec{w} \rangle|,$$

where $\vec{w} = (\partial F / \partial y, -\partial F / \partial x)|_p$.

(This last step is tricky. One way to see it is to note that adding a multiple of the function $\langle \text{grad } F, (x - x_0, y - y_0) \rangle$ to F doesn't change the Hessian, and leaves the curve β alone. So you can use the previous problem, where we assumed that $\text{grad } F = 0$)