Math 25b — Problem Set 7, due Friday, April 3

1. CS, p. 373 #14abde

2. CS, p. 399 #1bd, #2bf; p. 404 #1bd, #7

3. CS, p. 416 #7, #8

4. Suppose f_n is a sequence of integrable functions on a region $S \subset \mathbb{R}^n$, converging uniformly to a function f.

   (a) Show that f is integrable, and that

   $$\lim_{n\to\infty} \int_S f_n = \int_S f.$$

   (hint: if f between the functions $f_n + \epsilon$ and $f_n - \epsilon$, where f_n is integrable, then there are partitions for which the upper and lower sums of of f lie in the interval $(I_n - 2\mu(S)\epsilon, I_n + 2\mu(S)\epsilon)$, where $I_n = \int_S f_n$.)

   (b) Show by example that this result may not hold if the convergence is only assumed to be pointwise. (Three things may go wrong: f might not be integrable, the limit of $\int f_n$ might not exist, or the equality above might not hold. For this problem, just find $f_n \to f$ pointwise so that one of these fails)

5. Let $V_n(R)$ be the volume of the n-ball $B_n(R) = \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| \leq R\}$ of radius R. Show that these functions satisfy

   $$V_n(R) = \int_0^{2\pi}\int_0^R rV_{n-2}(\sqrt{R^2 - r^2})drd\theta,$$

   for $n > 2$. (Hint: use Fubini)

   Use this to get a formula for $V_n(R)$ (Use a change of variable; you will need to do the even and odd cases separately). Notice that for any fixed R, $\lim_{n\to\infty} V_n(R) = 0$!

6. In this problem we will show, more or less, that the shortest (smooth) path between two points is a straight line.

   (a) Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^1$ function whose derivative is also increasing. We call such a function *convex*. Show that for every $\lambda \in [0, 1]$, we have

   $$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y)$$

   (assume WLOG that $x < y$, let $b = \lambda x + (1-\lambda)y$, apply the mean value theorem to the intervals $[x, b]$ and $[b, y]$, and use the fact that ϕ' is increasing).

(b) Now suppose f is a continuous function on $[0, 1]$. Show that

$$\int_0^1 \phi(f(x))dx \geq \phi(t),$$

where $t = \int_0^1 f(x)dx$. Also show that equality occurs if and only if f is a constant function. (let $\beta = f'(t)$. Show that for any b we have $\phi(b) - \phi(t) \geq \beta(b - t)$. Let $b = f(x)$ and integrate with respect to x.)

(c) Show that $\phi(x) = \sqrt{1 + x^2}$ is a convex function. For $f \in \mathcal{C}^1[0, 1]$, we let

$$L(f) = \int_0^1 \sqrt{1 + (f'(x))^2} dx,$$

the length of the graph of f. Use part (b) to show that

$$L(f) \geq \sqrt{1 + (f(1) - f(0))^2},$$

and that equality holds if and only if f is linear.