Math 25b – Solution Set 7
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1. #14 (a) If $\exists m$: $\int_{S_m} f$ is not defined, then $\int_{R^n} f$ is not defined, as it suffices to take a partition that includes (or is a refinement of one that includes) S_m. Since $\inf U(f,P) > \sup L(f,P)$ on S_m (because it's not integrable), this will still be true if you add more rectangles. Now $I_m = \int_{S_m} f$ is a non-decreasing sequence, since f is non-negative, so it converges iff it is bounded.

(b) By the argument above, if $\int_S f$ is not defined, then it is not defined over any rectangle containing S. Since any sequence of nested rectangles whose union is the whole space is cofinal[1] in the set of all rectangles, ordered by inclusion[2], our sequence will eventually contain S, and if $\int_S f$ were not defined, neither would $\int_{\mathbb{R}^n} f$, which it is.

To show that this integral is independent of the sequence, we define

$$\int_{R^n} f = \sup_S \int_S f$$

where the supremum is taken over all rectangles in R^n. We now claim that this definition is equivalent to that given in the book. Why do we bother? This definition is canonical—no choices are involved (we don't pick a sequence)—and canonical definitions are nice for many reasons, one of which we see here. Now since any sequence is a subset of the set of all rectangles,

$$\sup_{\{S_m\}} \int_{S_m} f \leq \sup_S \int_S f$$

But since our sequence is cofinal, and the integral is non-decreasing under inclusion (since f is non-negative)

$$\sup_{\{S_m\}} \int_{S_m} f \geq \sup_S \int_S f$$

[1] We define a *partially ordered set* as a set S, together with a relation $\leq$ such that, $\forall x, y, z \in S$

(i) reflexive: $x \leq x$
(ii) anti-symmetric: $x \leq y$ and $y \leq x \implies x = y$
(iii) transitive: $x \leq y$ and $y \leq z \implies x \leq z$

Examples include the natural numbers (or integers, or rationals, or reals) with the standard $\leq$, sets with $\subseteq$ or partitions with refinement. The reason this is called a partial order is that not every two elements are comparable; you could have $x \not\leq y$ and $y \not\leq x$ (consider refinements). Now define $C \subset S$ to be *cofinal* in S if $\forall s \in S \exists c \in C$ such that $s \leq c$. To visualize this, consider the natural numbers with the standard order; the primes are cofinal in the natural numbers because given any number, there is a prime number greater or equal to it. Similarly, the square numbers are cofinal in the natural numbers. In terms of partitions, a set of partitions is cofinal in the set of all partitions if, given any partition, you can refine it into the cofinal set. Cofinal sets are useful because you can often (as in this problem) work just with a cofinal set, rather than the whole collection, which often makes computations and proofs much easier. You'll see them again if you go further in math, and they definitely take some getting used to.

[2] To see this, note that since $\cup_{m=1}^{\infty} S_m = \mathbb{R}^n$, this sequence will, in particular, cover any rectangle S. By compactness of S, there is a finite subcover, call it $\{S_{m_1}, \dots\}, S_{m_n}$. Now since the rectangles are nested, if we let $M = \max m_i$ then S_M will contain all the S_{m_i} and therefore S. So our sequence is cofinal.

(that is, given any $S, \exists S_m \colon S_m \supset S$ and $\int_{S_m} f \geq \int_S f$, so the above holds)
And lastly, since the integral is non-decreasing under inclusion,

$$\sup_{\{S_m\}} \int_{S_m} f = \lim_{m\to\infty} \int_{S_m} f$$

Combining the above, we obtain

$$\lim_{m\to\infty} \int_{S_m} f = \sup_S \int_S f$$

for any nested sequence whose union is the whole space, so in particular, our two limits agree.
The above discussion is much more than you would ever need to show, but it is hopefully an enlightening way of looking at this problem.

(d) For this part, you don't need to do a substitution to solve them. Note that $1 + x^2 \geq 1 - 2x + x^2 = (x-1)^2$ for $x \geq 0$, and $\frac{1}{1+x^2}$ is even, so

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx &= 2\int_0^{\infty} \frac{1}{1+x^2}\,dx \\ &\leq 2\int_0^{\infty} \frac{1}{(x-1)^2}\,dx \\ &= 2\int_1^{\infty} \frac{1}{x^2}\,dx = 2\end{aligned}$$

and since $\frac{1}{1+x^2}$ is continuous, non-negative, its integrals over any sequence of rectangles will be non-decrease, and bounded above, so it will converge, as desired.
Now $\frac{x}{1+x^2} \geq \frac{1}{x}$ for $x \geq 1$ and $\frac{x}{1+x^2} \geq 0$ iff $x \geq 0$. So $\int_{\mathbb{R}} f^+\,dx = \int_0^{\infty} f^+\,dx = \int_0^{\infty} f\,dx$ Now $k = \int_0^1 \frac{x}{1+x^2}\,dx$ is defined, because the function is continuous on a compact set (and in any case if it wasn't, we'd be done), so

$$\begin{aligned}\int_{\mathbb{R}} f^+\,dx &= \int_0^{\infty} f\,dx \\ &= k + \int_1^{\infty} \frac{x}{1+x^2}\,dx \\ &\geq k + \int_1^{\infty} \frac{1}{x}\,dx \\ &= k + \infty = \infty\end{aligned}$$

so the integral is not defined. The point is, you were not asked to evaluate these integrals, only to show convergence or divergence, so you can just compare them to easier functions, rather than using a table of integrals or a substitution.

(e) If $\int_{\mathbb{R}^n} f\,dx$ is defined, then so are $\int_{\mathbb{R}^n} f^+\,dx, \int_{\mathbb{R}^n} f^-\,dx$ and therefore so is

$$\int_{\mathbb{R}^n} |f|\ dx = \int_{\mathbb{R}^n} f^+ + f^-\,dx$$

since it is defined and bounded on all S.

2. Steps have been included only when instructive. (admire the increased legibility of putting spaces between the $dx\,dy$)

p. 399 # 1 (b) $\int_{-1}^{1}\int_{-x^2}^{x^2}(x^2 - y)\,dy\,dx = \frac{4}{5}$

(d) $\int_0^1\int_0^{2-2y}(x^2 + y^2)\,dx\,dy = \frac{5}{6}$

p. 399 # 2 (b) $\int_{S_2} x^2 + y^3 + 2xy\,dy\,dx = \int_{-1/\sqrt{2}}^{1/\sqrt{2}}\int_{x^2}^{1-x^2} x^2 + y^3 + 2xy\,dy\,dx = \frac{29\sqrt{2}}{140}$

(f) For this one, note that $(\log(1+y^2))' = 2y/(1+y^2)$. $\int_{S_1} \frac{y(1-x)}{1+y^2}\,dy\,dx = \frac{\log 4 - 1}{4}$

p. 404 # 1 (b) $\int_S x^2 + xy\,dy\,dx = \int_0^1\int_{x^3}^{x^2} x^2 + xy\,dy\,dx = 13/240$

(d)

$$\begin{aligned}
\int_S x\,dy\,dx &= \int_{-1}^{1}\int_{x^2-1}^{1-x^2} x\,dy\,dx \\
&= \int_0^1\int_{x^2-1}^{1-x^2} x\,dy\,dx + \int_{-1}^{0}\int_{x^2-1}^{1-x^2} x\,dy\,dx \\
&= \int_0^1\int_{x^2-1}^{1-x^2} x\,dy\,dx + -\int_0^1\int_{x^2-1}^{1-x^2} -x\,dy\,dx = 0
\end{aligned}$$

p. 404 # 7 This is much easier if you chose the order of integrals properly[3]. So we get

$$\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} 1\,dz\,dx\,dy = 4\int_{-3}^{3} 9 - y^2\,dy = 144$$

p. 417 # 7

$$Dh = \begin{pmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{pmatrix}$$

Expanding by minors on the last column yields $\det Dh = -\rho^2\sin\phi$, and by taking absolute values we obtain our result. Note that if you accidentally found the transpose of the matrix (like a previous version of this solution set), the determinant worked out—remember why?

p. 417 # 8 (a) This is a direct calculation, and $\int = \frac{2\pi}{5}(25\sqrt{5} - 1)$.

[3]and didn't immediately designate x as the *mystery* variable, and thus the last limit. Fight the logocentric linguistic hegemony!

(b) In spherical coordinates, one sees that this is a 1/12 wedge of the previous one, so $\int = \frac{\pi}{30}(25\sqrt{5}-1)$.

(c) $\int = \frac{\pi}{5}(2-\sqrt{2})(25\sqrt{5}-1)$

4. (a) As stated in the hint, this is an $\frac{\epsilon}{2}$ proof: we can get f_n arbitrarily close to f *at all points*, so we can get the upper and lower bounds (and therefore the integral) arbitrarily close. To make this formal, add symbols.

(b) Throughout, take $S = [0, 1]$. Naively, one might try

$$f_n(x) = \begin{cases} n & 0 \le x \le \frac{1}{n} \\ 0 & x > \frac{1}{n} \end{cases}$$

Unfortunately, this doesn't converge at 0, so instead we use

$$f_n(x) = \begin{cases} n & \frac{1}{n} \le x \le \frac{2}{n} \\ 0 & \text{otherwise} \end{cases}$$

Now $\forall x \lim_{n\to\infty} f_n(x) = 0$, so $f_n \to f = 0$ pointwise. However, $\int_0^1 f_n = 1 \neq 0 = \int_0^1 f$. If you want the integrals to diverge, you could use

$$g_n(x) = \begin{cases} n^2 & \frac{1}{n} \le x \le \frac{2}{n} \\ 0 & \text{otherwise} \end{cases}$$

Now $\forall x \lim_{n\to\infty} g_n(x) = 0$, so $g_n \to g = 0$ pointwise. However, $\lim_{n\to\infty} \int_0^1 g_n = \lim_{n\to\infty} n = \infty \neq 0 = \int_0^1 g$. Lastly, consider

$$h_n(x) = \begin{cases} \left(\frac{1}{q}\right)^{\frac{1}{n}} & x \in \mathbb{Q}, x = \frac{p}{q} \text{ in lowest terms} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Now by CS p. 352 #15 (or reasoning similar to it), h_n is integrable $\forall n$ with $\int_0^1 h_n = 0$. But $h = \lim_{n\to\infty} h_n$ is given by

$$h(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

because for $m > 0$, $\lim_{n\to\infty} m^{\frac{1}{n}} = 1$. But by CS p. 352 # 14, this is not integrable. And if you're really slick, you can take

$$h_n'(x) = \begin{cases} 1 & x \in \mathbb{Q},\ x = \frac{p}{q},\ q \le n \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Now $h_n' \to h$, and h_n' is non-zero for only finitely many points, so it's clearly integrable with integral zero. The point here is that pointwise convergence is very poorly behaved — points can jump arbitrarily far arbitrarily late in the sequence, and can move at various speeds.

5. This is one way to find to volume of the n-ball. If you'd like, I can show you several others (including a slick one using the Γ function).

   Apply a change of coordinates (to cylindrical):

   $$h(r,\theta,x_3,\ldots,x_n) = (r\cos\theta, r\sin\theta, x_3,\ldots,x_n)$$

   where $0 \le r \le R, 0 \le \theta \le 2\pi$. Now

   $$Dh = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}$$

   where I_n is the identity matrix in $\mathbb{R}^n$. Thus, $|\det Dh| = |r| = r$. Further,

   $$\|r\cos\theta, r\sin\theta\| = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} = r$$

   so for

   $$\|(r\cos\theta, r\sin\theta, x_3,\ldots,x_n)\| \le R$$

   we need $\|(x_3,\ldots,x_n)\| \le \sqrt{R^2-r^2}$. Thus, $V_n = \int_0^{2\pi}\int_0^R rV_{n-2}\left(\sqrt{R^2-r^2}\right)\,dr\,d\theta$. After working out a few examples, you will become convinced that the formula must be

   $$V_n = \begin{cases} R^n\pi^{n/2}\frac{1}{(n/2)!} & n \text{ even} \\ R^n\pi^{n/2}\frac{2^{(n+1)/2}}{\sqrt{\pi}n(!!)} & n \text{ odd} \end{cases}$$

   where $n(!!) = n(n-2)\ldots 1$ (this is pretty standard notation). This can be written more elegantly as

   $$V_n = \frac{R^n\pi^{n/2}}{\Gamma(1+(n/2))}$$

   where $\Gamma(n) = g(n-1)$ for g from problem set 8, # 5 (verify that these agree). To show these formulas, one simply proceeds by induction, starting with base cases $n=0, n=1$ or $n=2, n=3$, if you don't like degenerate balls. For $n=0$, we obtain $V_0 = 1$, since we have only the origin in B_0, and the measure on zero dimensions is to just count points (watch out for degenerate cases — sometimes they don't work quite right; indeed, the way that they are handled is often by trying to extend results like this e. g., $0! = 1$). For $n=1$, we obtain $V_1 = 2R$, since the length of $[-R,R]$ is $2R$. For $n=2,3$ we have the well-known formulas $V_2 = \pi R^2, V_3 = \frac{4}{3}\pi R^3$. The inductive step is direct, once you determine the formulas.

6. (a) We can assume, without loss of generality, $x < y$, since if $x = y$, we get $\phi(x) \le \lambda\phi(x) + (1-\lambda)\phi(x) = \phi(x)$ and for $x > y$ we simply switch $\lambda, (1-\lambda)$. By the

mean value theorem,

$$\exists c \in [x,b]\colon \phi'(c) = \frac{\phi(b)-\phi(x)}{b-x}$$
$$\exists d \in [b,y]\colon \phi'(d) = \frac{\phi(y)-\phi(b)}{y-b}$$

Note that the cases $b = x, b = y$ are trivially true. Now since ϕ' is non-decreasing, $\phi'(c) \leq \phi'(d)$ so

$$\begin{aligned}
\frac{\phi(b)-\phi(x)}{b-x} &\leq \frac{\phi(y)-\phi(b)}{y-b} \\
(y-b)(\phi(b)-\phi(x)) &\leq (b-x)(\phi(y)-\phi(b)) \\
\lambda(\phi(b)-\phi(x)) &\leq (1-\lambda)(\phi(y)-\phi(b)) \\
\phi(b) &\leq \lambda\phi(x) + 1-\lambda)\phi(y) \\
\phi(\lambda x + (1-\lambda)x) &\leq \lambda\phi(x) + (1-\lambda)\phi(y)
\end{aligned}$$

(b) $\phi(b) - \phi(t) \geq (b-t)\phi'(t)$ by the mean value theorem (divide by $(b-t)$ — for $b = t$ our assertion is trivial — and then use ϕ' non-decreasing, and $b-t$ positive or negative, respectively). We let $b = f(x)$ and integrate

$$\begin{aligned}
\int_0^1 \phi(f(x))\,dx - \int_0^1 \phi(t)\,dx &\geq \int_0^1 (f(x)-t)\phi'(t)\,dx \\
&= \phi'(t)\left(\int_0^1 f(x)\,dx - t\right) = 0
\end{aligned}$$

so by rearranging this last, $\int_0^1 \phi(f(x))\,dx \geq \int_0^1 \phi(t)\,dx = \phi(t)$.
Now if $f(x) = k$, then $t = \int_0^1 f(x)\,dx = k$, so

$$\int_0^1 \phi(f(x))\,dx = \int_0^1 \phi(k)\,dx = \phi(k) = \phi(t)$$

We will not show that equality implies $f(x) = k$ because, by the definition of increasing used in class, it doesn't—we need ϕ to be *strictly* convex. Consider $\phi(x) = x$, or even $\phi(x) = 0$. Then for any $f(x)$, $\int_0^1 \phi(f(x))\,dx = \phi\left(\int_0^1 f(x)\,dx\right)$. However, for the function we consider below, this condition holds. Here's intuitively why this is true: consider the tangent to $\phi(x)$ at t. For a strictly convex curve, this line will lie strictly below the curve (to visually, think of $\phi(x) = x^2$ and choose a point), and $\int_0^1 \phi(f(x))\,dx$ minus the integral along the tangent line measures how much f moves — and so if f ever moves, the integral of the difference will be positive. To visualize, consider $f(x) = 2x - 1$, so $t = 0$ and the integral that we keep referring to measures the distance between x^2 and the x-axis over $-1, 1$.

In symbols, for ϕ strictly convex

$$\int_0^1 \phi(f(x)) - [\phi(t) + \phi'(t)(f(x) - t)] \; dx \geq 0$$

$$\int_0^1 \phi(f(x))\, dx \geq \int_0^1 [\phi(t) + \phi'(t)(f(x) - t)] \; dx$$

with equality iff $f(x)$ is constant (this is geometrically obvious, so we won't prove it). However, the right side reduces (since everything is independent of x)

$$\int_0^1 \phi(t) + \phi'(t)(f(x) - t)\, dx = \phi(t) + \phi'(t) \left[\int_0^1 f(x)\, dx - t \right] = \phi(t)$$

so we obtain the desired result.

(c) $\phi'(x) = \frac{x}{\sqrt{1+x^2}}$. $x^2 \geq 0 \implies \frac{1}{\sqrt{1+x^2}} > 0 \implies \phi'(x) > 0$ for $x > 0$ so ϕ convex. $t = \int_0^1 f'(x)\, dx = f(1) - f(0)$ by the fundamental theorem of calculus. By part (b), $L(f) \geq \sqrt{1+t^2} = \sqrt{1 + (f(1) - f(0))^2}$. Further, equality holds iff f' is constant, i. e., iff f is linear.