Math 25b – Problem Set 9, due Friday, April 24.

1. CS, p. 491 #2bdf, #3, #5
2. CS, p. 499 #2bdf, #3 (for the forms in #2bdf), #5, #8
3. CS, p. 504 #4acdg #6 (for the forms in #4acdg) p. 512 #2bd
4. Let $\Lambda_n^k \subset \Omega^k(\mathbb{R}^n)$ be the finite dimensional vector space of forms $\sum c_J \; dx_{j_1} \dots dx_{j_n}$ where the c_J are constants.

   (a) Show that Λ_n^k is the space of k-forms ω so that for all $\vec{v} \in \mathbb{R}^n$ we have $\phi_v^* \omega = \omega$, where $\phi_{\vec{v}}(\vec{x}) = \vec{v} + \vec{x}$ is a translation of $\mathbb{R}^n$. (we say that elements of Λ_n^k are *invariant under translations*)

   (b) Show that the multiplication of differential forms gives a nondegenerate bilinear pairing

   $$\Lambda_n^k \times \Lambda_n^{n-k} \to \Lambda_n^n,$$

   "nondegenerate" means that if $\eta\omega = 0$ for all η, then $\omega = 0$.

   (c) Take a nonzero $\eta \in \Lambda_n^1$, and show that for any $\omega \in \Lambda_n^k$, we have $\eta\omega = 0$ if and only if $\omega = \eta\mu$ for some $\mu \in \Lambda_n^{k-1}$

   (For "only if", first check it for $\eta = dx_1$, and then for general η find an invertible linear transformation T on $\mathbb{R}^n$ so that $T^*\eta = dx_i$)

   (d) Take $\omega \in \Lambda_3^2$. Prove that there are $\eta_1,\, \eta_2 \in \Lambda_n^1$ so that $\omega = \eta_1\eta_2$.

   (The map $\Lambda_3^1 \to \Lambda_3^3$ given by multiplying by ω on the right must have a nontrivial kernel; let η_1 be a nonzero element of the kernel, and use part (c).)

   (e) Take $\omega \in \Lambda_4^2$, and let $T\colon \Lambda_4^1 \to \Lambda_4^3$ be multiplication by ω on the right.

   If ω is a product of two elements of Λ_4^1, show that $\operatorname{rank} T = 2$. Use this to show that $dx_1 dx_2 + dx_3 dx_4$ doesn't factor into a product of two 1-forms.