Math 25b – Problem Set 10, due Friday, May 1.

Reminder: the final exam is Thursday, May 14 at 9:15am in Science Center E.

1. CS, p. 523 #3, #5

2. We will use the winding number $w(\gamma)$ to give another proof of the fundamental theorem of algebra. Let $n > 1$ and take a polynomial

$$f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$$

with complex coefficients. Suppose that $f(z) \neq 0$ for all $z \in \mathbb{C}$.

   (a) Show that there is an $r_0 > 0$ so that for all $z \in \mathbb{C}$ with $|z| = r_0$ we have $|z^n| > |f(z) - z^n|$.

   (b) For each $r \geq 0$ define a closed path $\gamma_r \colon [0,1] \to \mathbb{C} \setminus \{0\}$ by

$$\gamma_r(t) = f(re^{(2\pi t)i}) = f(r\cos(2\pi t) + ir\sin(2\pi t)).$$

   (Recall the definition of $e^{i\theta}$ from exercise 4, p. 307 of CS)
   We can identify $\mathbb{C} \setminus \{0\}$ with $\mathbb{R}^2 \setminus \{(0,0)\}$. Show that the winding number $w(\gamma_r)$ doesn't depend on r.

   (c) Show that γ_{r_0} is homotopic to the path given by $\gamma(t) = e^{(2n\pi t)i}$, and that $w(\gamma) = n$. Thus $w(\gamma_0) = n$, a contradiction.

3. Let $f(x,y)$ be a $\mathcal{C}^2$ function on $\mathbb{R}^2$. Suppose that f is *harmonic;* this means that the Laplacian

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

is zero everywhere.

   (a) Define a path $\gamma_r(t) = (r\cos t, r\sin t)$ for $t \in [0, 2\pi]$, $r > 0$. Define a function g by

$$g(r) = \frac{1}{2\pi}\int_0^{2\pi} f(\gamma_r(t))\,dt$$

   Differentiate under the integral sign to get

$$g'(r) = \frac{1}{2\pi r}\int_{\gamma_r} -\frac{\partial f}{\partial y}\,dx + \frac{\partial f}{\partial x}\,dy$$

   (b) Use Stokes' theorem to show that $g'(r) = 0$, and hence

$$g(r) = f(0,0)$$

   Thus the average value of a harmonic function f on a circle with center p is $f(p)$.

4. (The volume form on a surface). Let $S \subset \mathbb{R}^3$ be a smooth surface defined by the equation $f = 0$. Define a vector field $\vec{n}\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $\vec{n}(p) = \operatorname{grad} f / \|\operatorname{grad} f\|$. Then for $p \in S$, $\vec{n}(p)$ is a unit vector perpendicular to T_pS.

   Let dA be the 2-form on $\mathbb{R}^3$ defined by $\vec{n}$; i.e. let $dA = n_1\, dy\, dz - n_2\, dx\, dz + n_3\, dx\, dy$. We define the surface area of S to be $|\int_S dA|$.

   (a) Let $S = S_r$ be the sphere of radius r centered at the origin; use the equation $x^2 + y^2 + z^2 - r^2 = 0$ to define S. Using the divergence theorem, show that the surface area of S is $4\pi r^2$.

   (b) Do the same calculation by parametrizing S in spherical coordinates: let $h(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$ and calculate $h^* dA$ explicitly.

   (c) Show that the form $\omega = 1/r^2 dA$ (where $r^2 = x^2 + y^2 + z^2$) satisfies $d\omega = 0$. Use the previous parts of this problem to show that there is no 1-form η with $d\eta = \omega$. This form is often called the "solid angle form" in $\mathbb{R}^3$.