Math 25b — Problem Set 11, for review (not due)

Reminder: the final exam is Thursday, May 14 at 9:15am in Science Center E. There will be a review session on Monday May 11th at the usual place and time.

1. Let γ be a piecewise $\mathcal{C}^1$ path which first travels the quarter circle centered at the origin from $(1,0)$ to $(0,1)$ counterclockwise, and then goes in a straight line from $(0,1)$ to $(-1,0)$. Calculate the integral $\int_\gamma 2xy\,dx - y^2\,dy$, first by direct calculation, and second by using Green's theorem to relate this to an integral over a simpler path.

2. Let Γ be the curve in $\mathbb{R}^3$ which is the intersection of the cylinder $x^2+y^2=4$ and the surface $z=xy$, oriented counterclockwise when viewed from above. Compute the line integral $\int_\Gamma (3y\,dx + yz\,dy + (e^{\sin z} - x^2/2)\,dz)$, first by pulling back the integral to $\mathbb{R}^2$ and using Green's theorem, and then by using Stoke's theorem directly (what is a suitable surface whose (oriented) boundary is Γ, and how would you parametrize it?).

3. Consider the conical surface $S = \{(x,y,z) \in \mathbb{R}^3 \mid 4(x^2+z^2) = (2-y)^2, 0 \le y \le 2\}$. Orient S so the normal vector has positive second coordinate at each point.
   (a) Find a parametrization for S and ∂S. Be careful about orientations.
   (b) Find $\int_S d(x^3\,dx + y^3\,dy - x^3\,dz)$ in as many ways as possible, including by the trick in the next part. (A helpful formula: $\cos^4\theta + \sin^4\theta = (3+\cos 4\theta)/4$.)
   (c) Find $\int_S y\,dx\,dy + (xz+y)dx\,dz + yz\,dy\,dz$. *Hint:* What is the integral over S plus the flat disc with boundary ∂S?

4. (a) Show that the set S of $n\times n$ real matrices which are diagonalizable and have all eigenvalues different is an open subset of the space $\mathcal{M}(n\times n)$ of all matrices.
   (b) Suppose a matrix A satisfies $AB = BA$ for some $B \in S$. Show that A is diagonalizable. Is A in S?

5. Let f and g be nonnegative continuous functions on $I = [0,1]$, and suppose that $f(x)g(x) \ge 1$ for all $x \in I$. Show that $(\int_I f)(\int_I g) \ge 1$. *Hint:* $\langle f,g\rangle = \int_0^1 fg$ defines an inner product on $\mathcal{C}([0,1])$.

6. Put an inner product on $\mathcal{P}_n$ by $\langle f,g\rangle = \int_{-1}^1 fg$ (why is this positive definite?). Consider the linear transformation $T\colon \mathcal{P}_n \to \mathcal{P}_n$ defined by

$$T(f) = \frac{d}{dx}\left(\left(1-x^2\right)\frac{df}{dx}\right)$$

   (a) Show that T is self-adjoint. What are its eigenvalues?
   (b) Show that for any $k \le n$ there is a unique monic[1] eigenvector of degree k; i. e., of the form

$$p_k(x) = x^k + a_{k,k-1}x^{k-1} + \cdots + a_{k,0}.$$

[1] Leading coefficient 1

(c) Show that the different p_k are orthogonal. What is the eigenvalue of p_k?

(d) What fails if we try to do the same thing with $T(p) = d^2p/dx^2$?

7. (a) Show that the limit $\lim_{t\to\infty} \int_1^t \frac{\sin x}{x}\, dx$ exists (integrate by parts)

(b) Show that $\lim_{k\to\infty} \sum_{n=1}^k 1/n$ is infinite, by comparing the integral $\int_1^\infty dx/x$ with the integral of a step function on $[1,\infty)$.

(c) Show that the improper integral $\int_1^\infty \frac{\sin x}{x}$ does not exist with the definition we have been using. We say that the integral converges, but does not converge absolutely (i. e., $|\sin x/x|$ can't be integrated on $[1,\infty)$).
Hint: Show that

$$\int_{2k\pi}^{2(k+1)\pi} \left|\frac{\sin x}{x}\right| \geq \frac{2}{(2k+1)\pi}$$

and use (b).

(d) Prove the following related result about infinite series. Let a_n be a *decreasing* sequence of *nonnegative* numbers, and suppose that $\lim_{n\to\infty} a_n = 0$. Show that

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges, i. e., the sequence of partial sums converges.
Hint: Define two sequences $e_n = \sum_{k=1}^{2n} (-1)^k a_k$ and $o_n = \sum_{k=1}^{2n-1} (-1)^k a_k$. Show that e_n is a decreasing sequence, o_n is an increasing sequence, and both are bounded; thus they both have limits. Look at $e_n - o_n$ and conclude that the limits are equal.

8. (a) Let $f\colon [0,1] \to \mathbb{R}$ be an integrable function, and let P_n be the partition of $[0,1]$ into n equal parts: $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\right\}$. Let $l_n = L(f, P_n)$ and $u_n = U(f, P_n)$, and $a_n = \frac{1}{n}\sum_{k=1}^n f(\frac{k}{n})$. Show that all three of these sequences converge to $\int_{[0,1]} f$.
Hint: Given $\epsilon > 0$, find a partition P for which $U(f,P) - L(f,P) < \epsilon/2$. Then find an n so that $u_n - l_n$ is close to $U(f,P) - L(f,P)$.

(b) Give an example of a nonintegrable function f for which the three sequences converge but have different limits.

(c) Why didn't we define the integral to be the common limit of the sequences $\langle l_n \rangle$ and $\langle u_n \rangle$, when they are equal? Think about what would happen to the proofs of the basic properties of the integral.

9. Let $T\colon \mathbb{C}^n \to \mathbb{C}^n$ be an invertible linear transformation for which T^2 is diagonalizable. Show that T is diagonalizable.

Hint: if $\vec{v}$ is an eigenvector for T^2 but not for T, show that there are two independent eigenvectors in the span of $\vec{v}$ and $\vec{w} = T\vec{v}$.

10. (Matrix exponentiation) Let $M = \mathcal{M}(n \times n)$ be the vector space of all $n \times n$ real matrices. Define a function $\exp\colon M \to M$ by

$$\exp(A) = \operatorname{Id} + A + \frac{A^2}{2} + \cdots + \frac{A^n}{n!} + \cdots$$

i. e., the limit of the sequence of partial sums. Show that this converges for any A (you may find it useful to know that there is a norm on M for which $\|AB\| \leq \|A\| \, \|B\|$). For $n = 1$ this agrees with the usual exponential function $\exp(a) = e^a$.

(a) If B is an invertible matrix, show that $\exp(BAB^{-1}) = B \exp(A) B^{-1}$.

(b) If A is diagonalizable, show that $\det(\exp A) = e^{\operatorname{tr} A}$. This is actually true for any matrix.

(c) Calculate $\exp\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$.

(d) Suppose A and B are diagonalizable, and $AB = BA$. Show that $\exp(A + B) = \exp(A)\exp(B)$. Find diagonalizable matrices A and B for which the formula fails.

(e) If A is a skew-symmetric matrix[2], show that $\exp(A)$ is orthogonal.

[2] $A = -A^t$