MATH 25A – PROBLEM SET #4
FRIDAY OCTOBER 29

1. Part A

Definition. Let S be a nonempty set. A *topology* on S is a collection τ of subsets of S satisfying the following properties:

a. $S, \emptyset \in \tau$,
b. $\cup_{i \in I} U_i \in \tau$ if all $U_i \in \tau$ (here I is an index set, possibly infinite),
c. $\cap_{i=1}^{m} U_i \in \tau$ if all $U_i \in \tau$.

The subsets $U_i \in \tau$ are called *open*. A subset is *closed* if its complement is open. A set S with a topology τ is called a *topological space.*

Definition. A map $S \to T$ between topological spaces is *continuous* if $f^{-1}(U)$ is open in S whenever U is open in T.

1. Prove that the following pairs (S, τ) are topological spaces:
   (a) (Trivial topology) Let $\tau = \{\emptyset, S\}$.
   (b) (Discrete topology) Let τ consist of all subsets of S.
   (c) (Metric topology) Let (S, d) be a metric space, and define U to be in τ if for any $a \in U$ there exists $\epsilon > 0$ such that the open ϵ-ball $B_\epsilon(a)$ lies in U. (Here open balls are defined with respect to the metric d).
   (d) (Subspace topology) Let (T, σ) be a topological space, S a nonempty subset of T, and define $U \subset S$ to be in τ if $U = V \cap S$ for some $V \in \sigma$.
   (e) (Zariski topology) Let $S = \mathbb{R}^n$ and define $U \in \tau$ if the complement of U is the locus where a finite set of polynomials $p_i \in \mathbb{R}[X_1, \dots, X_n]$ vanishes:
   $$\mathbb{R}^n - U = \{x \in \mathbb{R}^n \,|\, p_1(x) = p_2(x) = \dots = p_m(x) = 0\}.$$
   (You have to use Hilbert Basis Theorem: if a set C in $\mathbb{R}^n$ is defined by the vanishing of an infinite number of polynomial equations then one can choose from this infinite set of polynomials a finite number with the same vanishing locus.)

2. (a) Prove that two equivalent metrics on the same set S define the same topology.
   (b) Show that every set in $\mathbb{R}^n$ that is open in the Zariski topology is also open in the usual topology. (You can assume that functions defined by polynomials are continuous in the usual sense.) Show that the converse is not true.

3. Let $S = \mathbb{N} \cup \{\infty\}$, and let the topology on S be
$$\tau = \{U \subset \mathbb{N}\} \cup \{U \subset S \,|\, S - U \quad \text{is finite}\}.$$

   (a) Prove that this defines a topology on S.
   (b) We consider a sequence (a_i) in $\mathbb{R}^n$ as a map $f_a : \mathbb{N} \to \mathbb{R}^n$, where $f_a(i) = a_i$. Prove that the sequence converges if and only if the map f_a can be extended to a continuous map $\overline{f}_a : S \to \mathbb{R}^n$. That means, $\overline{f}_a|_\mathbb{N} = f_a$.

4. (a) Show that a map $f : \mathbb{R}^n \to \mathbb{R}^m$ defined by:

$$f(x) = \begin{pmatrix} q_1(x) \\ q_2(x) \\ \vdots \\ q_m(x) \end{pmatrix}$$

for some polynomials $q_i \in \mathbb{R}[X_1, \ldots , X_n]$ is continuous if both $\mathbb{R}^n$ and $\mathbb{R}^m$ are given the Zariski topology.

(b) Show that the converse is not true: there are non-polynomial maps that are continuous in the Zariski topology. (Hint: find such a map $f : \mathbb{R} \to \mathbb{R}$.)

2. PART B.

Definition. A subset K in a topological space S is *compact* if every open cover of K has a finite subcover.

Definition. A subset V of a topological space S is *connected* if for any two disjoint open sets U_1 and U_2 in S that cover V:

$$V \subset U_1 \cup U_2$$
$$U_1 \cap U_2 = \emptyset,$$

we have that either $V \subset U_1$ or $V \subset U_2$.

1. (a) What are the compact sets if S has the discrete topology.
   (b) Write the definition of a compact set in terms of closed sets: For every infinite collection of closed sets in S satisfying ..., one can choose a finite sub-collection satisfying ... .
   (c) Prove that $\mathbb{R}^n$ with Zariski topology is compact. (Hint: use the previous problem and Hilbert Basis Theorem.)
   (d) Prove that if $K \subset S$ is compact and $L \subset S$ closed such that L is containd in K, then L is also compact.
2. If $f : S \to T$ is a continuous map and $C \subset S$ a connected set, show that $f(C)$ is also connected.
3. Define a topology on $\mathbb{R}$ as follows: a set U is open if and only if for every point $a \in U$ there exists a half-open interval $[a, a + \epsilon) \subset U$ for some $\epsilon > 0$.
   (a) Show that this defines a topology.
   (b) Show that $\mathbb{R}$ is not connected in this topology.
   (c) Show that $[0, 1]$ is not compact in this topology.