MATH 25A – PROBLEM SET #6
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1. Part A

1. Problem 2.2.8 in the book.
2. Problem 2.3.2 in the book.
3. (The meaning of row reduction) Let V and W be vector spaces with bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_m\}$, respectively, and let $L : V \to W$ be a linear map represented by a matrix A in these bases: $L(v_j) = \sum_i w_i a_{ij}$.
   (a) Show that a row operation corresponds to changing the basis of W. That means, if we get A' from A by a row operation, then A' represents the map L with respect to a new basis $\{w'_1, \ldots, w'_m\}$ of W. For example, multiplying the i'th row by λ corresponds to replacing w_i by $\frac{1}{\lambda} w_i$.
   (b) Show that a matrix $\tilde{A}$ in echelon form corresponds to a very special choice $\{\tilde{w}_1, \ldots, \tilde{w}_m\}$ of a basis for W. Here is how the construction starts: Suppose L maps the first $i_1 - 1$ basis elements $v_1, \ldots, v_{i_1 - 1}$ to zero, but $L(v_{i_1}) \neq 0$. Then take $\tilde{w}_1 = L(v_1)$. Now suppose L maps the next $i_2 - i_1 - 1$ basis elements $v_{i_1+1}, \ldots, v_{i_2-1}$ to multiples of $\tilde{w}_1$, but $L(v_{i_2})$ is linearly independent from $\tilde{w}_1$. Then take $\tilde{w}_2 = ...$
   (c) Is the basis $\{\tilde{w}_1, \ldots, \tilde{w}_m\}$ in (b) unique? Does it contain a basis for the image of L?
   (d) Show that a matrix A row-reduces to a unique matrix $\tilde{A}$ in echelon form.

2. Part B.

1. Problem 2.4.12 in the book.
2. Let V be a finite dimensional vector space. Prove that $\{v_1, \ldots, v_n\}$ is a basis of V if and only if for any vector space W and any n vectors $w_1, \ldots, w_n \in W$, there exists a unique linear map $L : V \to W$ such that
$$L(v_i) = w_i \qquad i = 1, \ldots, n.$$

3. Part C.

1. Find a basis for $Mat(m, n)$. What is the dimension of $Mat(m, n)$.
2. Given vector spaces V and W with bases $\{v_1, \ldots, v_n\}$ and $\{w_j\}_{j \in J}$, respectively, find a basis for $Hom(V, W)$. Recall that $Hom(V, W)$ is the vector space of all linear transformations from V to W. (Hint: in question 1, what is an element of the basis of $Mat(m, n)$ as a linear map? Extend this idea to the present case. But note that J may be infinite.)
3. Does your proof for the previous problem work when V is infinite dimensional? For example, consider $L \in Hom(\mathbb{R}[X], \mathbb{R})$ given by
$$p(X) \mapsto \int_0^1 p(X) dX.$$
   Can L be written as a finite linear combination of basis elements constructed in the previous problem?
4. Problem 2.4.6 (b),(c),(d) in the book.

4. Part D

Problem 2.5.16 in the book.