MATH 25A – PROBLEM SET #8
FRIDAY DECEMBER 10

1. Part A

Let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable function (hence bounded, with bounded support). Define S to be the region under the graph of f:

$$S = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} | 0 \leq y \leq f(x)\}.$$

Prove that S is pavable and

$$Vol_{n+1} S = \int_{\mathbb{R}^n} f d^n x.$$

Note that the volume of S is defined by the integral of the characteristic function of S. You can compare the two integrals by showing that the L_N's and U_N's are very close to each other.

2. Part B.

1. Find the area bounded by the ellipse
$$(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1.$$
2. Problem 4.5.18 in the book.

3. Part C.

Example 4.5.6 in the book describes how to compute the volume of a ball in $\mathbb{R}^n$. Read this example and do problems 4.5.4, 4.5.5.

4. Part D.

1. Let $f : [a, b] \to \mathbb{R}$ be a monotonely increasing function:
$$f(x) \leq f(y) \qquad \text{if } a \leq x \leq y \leq b.$$
Show that f is integrable. (Hint: how many intervals $C \in D_N$ can have $osc_C(f) > \epsilon$?)
2. (Compare with Example 4.4.3 in the book) Define $f : (0, 1) \to \mathbb{R}$ by
$$f(x) = \begin{cases} \frac{1}{q^3} & \text{if } x = \frac{p}{q} \text{ is rational, written in lowest terms} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$
   (a) Prove that f is continuous at all irrational points and discontinuous at all rational points. Hence the set Δ where f is not continuous does not have volume zero.
   (b) Define
   $$g(x) = \sum_{\frac{p}{q} \leq x} \frac{1}{q^3}, \qquad h(x) = \sum_{\frac{p}{q} < x} \frac{1}{q^3},$$
   where the sums run over all rational numbers $\frac{p}{q} \in (0, x]$ (respectively, $\frac{p}{q} \in (0, x)$), written in the lowest terms. Prove that these functions are well-defined, that means the sums converge for all x.
   (c) Prove that g and h are integrable.
   (d) Prove that $f = g - h$, hence f is integrable.