MATH 25A – PRACTICE EXAM #1

(1) Let $A \in Mat(m,n)$ and $B \in Mat(n,p)$. Prove that

$$rank(AB) \leq rank(A)$$
$$nullity(AB) \geq nullity(B)$$

(2) A matrix $O \in Mat(n,n)$ is called orthogonal if its columns form an orthonormal basis of $\mathbb{R}^n$.
   (a) If O is an orthogonal matrix, prove that O^T is its inverse.
   (b) Let A be a symmetric matrix. Prove that there exists an orthogonal matrix O such that

$$O^{-1}AO$$

   is diagonal (that means, the only nonzero entries lie on the diagonal).
   (c) Use part (b) to find A^{100} for

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

   (d) Let A be a symmetric matrix. Use part (b) to prove that A is positive definite if and only if there exists an invertible matrix W such that $A = W^T W$. (Hint: start with the case where A is diagonal.)

(3) Define a sequence $s_1 = \sqrt{2}$, $s_{n+1} = \sqrt{2 + s_n}$ for $n \geq 1$. Prove that s_n converges to a limit $s \leq 2$.

(4) Let $f : [a,b] \to \mathbb{R}$ be a function.
   (a) Give the definition for f being continuous in terms of converging sequences.
   (b) Prove that f is continuous if and only if any sequence $\{x_i\}$ in $[a,b]$ has a subsequence x_{i_j} converging to some $x \in [a,b]$, such that $f(x_{i_j})$ converges to $f(x)$.
   (c) Prove that f is continuous if and only if the graph of f

$$G(f) = \{(x, f(x)) \in \mathbb{R}^2 | x \in [a,b]\}$$

   is compact.

(5) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(0) = 0$ and

$$f(x,y) = \frac{xy^2}{x^2 + y^4}$$

for $(x,y) \neq 0$.
   (a) Prove that f restricted to any line in $\mathbb{R}^2$ is continuous.

(b) Prove that f is not continuous at 0. (Hint: do not use polar coordinates. Instead, consider f restricted to a curve of the form $x = y^k$.)

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that

$$\lim_{x\to\infty} f'(x) = M$$

for some M. Prove that

$$\lim_{x\to\infty} f(x+1) - f(x) = M$$

(7) This problem gives an inequality between the geometric mean and the arithmetic mean of non-negative numbers.

(a) Find the maximum of $x_1^2 \cdot \ldots \cdot x_n^2$ subject to the condition $x_1^2 + \ldots + x_n^2 = 1$.

(b) Prove that $(x_1^2 \cdot \ldots \cdot x_n^2)^{1/n} \leq 1/n$ if $x_1^2 + \ldots + x_n^2 = 1$.

(c) Prove that

$$(a_1 \cdot \ldots \cdot a_n)^{1/n} \leq \frac{a_1 + \ldots + a_n}{n}$$

for $a_i > 0$.

(8) Let C be a circle on the Earth's surface (e.g., a meridian). Prove that at any given moment there exist two points on C with equal temperature. (You may assume that temperature is a continuous function.)

(9) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$ and $\lim_{x\to\infty} f(x) = 0$.

(a) Give an exact definition of the property $\lim_{x\to\infty} f(x) = 0$.

(b) Prove that there exists a point $a \in \mathbb{R}$ such that $f'(a) = 0$.

(10) Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a continuously differentiable function satisfying the Cauchy-Riemann equations:

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y}, \qquad \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x}$$

(a) Prove that $Df(a)$ is invertible if and only if $Df(a) \neq 0$.

(b) If $Df(a) \neq 0$ then f has an inverse near a. Prove that the inverse also satisfies the Cauchy-Riemann equations.