MATH 25A – PRACTICE EXAM #1

(1) Let $A \in Mat(m,n)$ and $B \in Mat(n,p)$. Prove that

$$rank(AB) \le rank(A)$$
$$nullity(AB) \ge nullity(B)$$

Proof. Consider A and B as linear maps and their product as a composition. Then $Image(AB) \subset Image(A)$ and $Ker(B) \subset Ker(AB)$. The two inequalities follow from this.

(2) A matrix $O \in Mat(n,n)$ is called orthogonal if its columns form an orthonormal basis of $\mathbb{R}^n$.

(a) If O is an orthogonal matrix, prove that O^T is its inverse.
Proof. It is easy to see that $O^T O = I$. This actually implies the other equality $OO^T = I$ the same way as we proved in class.

(b) Let A be a symmetric matrix. Prove that there exists an orthogonal matrix O such that

$$O^{-1}AO$$

is diagonal (that means, the only nonzero entries lie on the diagonal).
Proof. By the spectral theorem A has a set of eigenvectors that forms an orthonormal basis of $\mathbb{R}^n$. Put these vectors as columns of O.

(c) Use part (b) to find A^{100} for

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

Proof. The first step is to find O as in previous problem. For this we have to find an orthonormal basis of eigenvectors. This can be done simply by solving the equation $Av = \lambda v$ for λ and v. Now the matrix $O^{-1}AO$ is diagonal, so it is very easy to multiply it with itself 100 times. The result is almost A^{100}.

(d) Let A be a symmetric matrix. Use part (b) to prove that A is positive definite if and only if there exists an invertible matrix W such that $A = W^T W$. (Hint: start with the case where A is diagonal.)
Proof. One direction is easy: if $A = W^T W$ then A is positive definite. For the other direction let's assume that A is positive definite. First, if A is diagonal, one can let W be the diagonal matrix $W_{ij} = \sqrt{A_{ij}}$. We can take square roots because the diagonal entries of A are positive by positive definiteness (check $e_i^T A e_i > 0$). Then since W has nonzero entries on the diagonal, it is invertible. For the general A, use part (b) to construct a diagonal matrix $B = O^{-1}AO$, and check that this is positive

definite. Then we can write $B = W_1^T W_1$ for some invertible W_1. Finally express A in terms of B.

(3) Define a sequence $s_1 = \sqrt{2}$, $s_{n+1} = \sqrt{2 + s_n}$ for $n \geq 1$. Prove that s_n converges to a limit $s \leq 2$.

Proof. It can be shown by induction that the sequence s_n is increasing and bounded from above by 2.

(4) Let $f : [a, b] \to \mathbb{R}$ be a function.

(a) Give the definition for f being continuous in terms of converging sequences.

Definition. f is continuous at $x \in [a, b]$ if for any sequence x_i in $[a, b]$ converging to x, the sequence $f(x_i)$ converges to $f(x)$.

(b) Prove that f is continuous if and only if any sequence $\{x_i\}$ in $[a, b]$ has a subsequence x_{i_j} converging to some $x \in [a, b]$, such that $f(x_{i_j})$ converges to $f(x)$.

Proof. First assume that f is continuous and $\{x_i\}$ a sequence in $[a, b]$. Since $[a, b]$ is compact, there is a subsequence converging to some $x \in [a, b]$. Now apply the definition of the continuity to this subsequence.

Conversely, assume that f is not continuous at some $x \in [a, b]$. Then for some $\varepsilon > 0$ there exists no $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. Use this to construct a sequence $\{x_i\}$ converging to x such that f applied to any subsequence can not converge to $f(x)$.

(c) Prove that f is continuous if and only if the graph of f

$$G(f) = \{(x, f(x)) \in \mathbb{R}^2 \,|\, x \in [a, b]\}$$

is compact.

Proof. Assume that f is continuous. If $\{(x_i, f(x_i))\}$ is any sequence in $G(f)$, we can use the compactness of $[a, b]$ to choose a subsequence $\{x_{i_j}\}$ converging to some $x \in [a, b]$. By continuity of f, the sequence $\{f(x_{i_j})\}$ converges to $f(x)$. From this we get that the subsequence $\{(x_{i_j}, f(x_{i_j}))\}$ converges to $(x, f(x)) \in G(f)$, hence $G(f)$ is compact.

Conversely, assume that $G(f)$ is compact. We want to prove that f satisfies the condition of (b). If $\{x_i\}$ is any sequence in $[a, b]$, consider the sequence $\{(x_i, f(x_i))\}$ in $G(f)$. By compactness of $G(f)$ we can choose a convergent subsequence and prove the condition in (b). (However, be careful to show that $f(x_{i_j})$ converges to $f(x)$ and not just any number.)

(5) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(0) = 0$ and

$$f(x, y) = \frac{xy^2}{x^2 + y^4}$$

for $(x, y) \neq 0$.

(a) Prove that f restricted to any line in $\mathbb{R}^2$ is continuous.

Proof. f is clearly continuous away from zero, hence its restriction to any line that does not pass through zero is also continuous. Consider a line through zero, for example given by $y = ax$ for some a. Substituting this into the formula of f we get a function of one variable, depending on a parameter a. The only place where this function is not continuous is where the denominator vanishes.

(b) Prove that f is not continuous at 0. (Hint: do not use polar coordinates. Instead, consider f restricted to a curve of the form $x = y^k$.)
Proof. Consider the curve $x = y^2$.

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that

$$\lim_{x\to\infty} f'(x) = M$$

for some M. Prove that

$$\lim_{x\to\infty} f(x+1) - f(x) = M$$

Proof. Apply the mean value theorem to $f(x+1) - f(x)$.

(7) This problem gives an inequality between the geometric mean and the arithmetic mean of non-negative numbers.
(a) Find the maximum of $x_1^2 \cdot \ldots \cdot x_n^2$ subject to the condition $x_1^2 + \ldots + x_n^2 = 1$.
(b) Prove that $(x_1^2 \cdot \ldots \cdot x_n^2)^{1/n} \leq 1/n$ if $x_1^2 + \ldots + x_n^2 = 1$.
(c) Prove that

$$(a_1 \cdot \ldots \cdot a_n)^{1/n} \leq \frac{a_1 + \ldots + a_n}{n}$$

for $a_i > 0$.

Proof. Use Lagrange multipliers to find the maximum. For the last part, let $x_i = \sqrt{a_i}$ scaled by a constant.

(8) Let C be a circle on the Earth's surface (e.g., a meridian). Prove that at any given moment there exist two points on C with equal temperature. (You may assume that temperature is a continuous function.)

Proof. There is a continuous map from the interval $[0, 1]$ onto the circle, mapping the endpoints 0 and 1 to the same point. Consider the composition of this map with the temperature function: we get a continuous function on $[0, 1]$ taking the same value at the endpoints. Now we can find two points in $[0, 1]$, distinct from the endpoints, where the function takes the same value.

(9) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$ and $\lim_{x\to\infty} f(x) = 0$.
(a) Give an exact definition of the property $\lim_{x\to\infty} f(x) = 0$.
Definition We say that $\lim_{x\to\infty} f(x) = L$ if for any $\varepsilon > 0$ there exists a $M > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > M$.
(b) Prove that there exists a point $a \in \mathbb{R}$ such that $f'(a) = 0$.

Proof. Consider f defined for non-negative numbers x only. It suffices to show that f has a global maximum or a global minimum point $a \in (0, \infty)$; then $f'(a) = 0$. Suppose there exists $b \in (0, \infty)$ such that $f(b) \neq 0$ (If $f(x) = 0$ for any x then the claim is trivial). Assume that $f(b) > 0$ and look for the global maximum of f. We can find $M > 0$ such that $|f(x)| < f(b)$ whenever $x > M$. Now a global maximum of f in $[0, M]$,which exists by compactness of $[0, M]$, must also be a global maximum of f in $[0, \infty)$. (Or similarly, on the interval $[0, M]$ the function f must achieve its maximum at some interior point $a \in (0, M)$, for which $f'(a) = 0$.)

(10) Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a continuously differentiable function satisfying the Cauchy-Riemann equations:

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y}, \qquad \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x}$$

(a) Prove that $Df(a)$ is invertible if and only if $Df(a) \neq 0$.

(b) If $Df(a) \neq 0$ then f has an inverse near a. Prove that the inverse also satisfies the Cauchy-Riemann equations.

Proof. One can write out explicitly the inverse of a 2×2 matrix. In particular, a matrix is invertible if and only if its determinant is nonzero. The derivative matrix of the inverse is the inverse of the derivative, hence we can read off the partial derivatives of the inverse function from the inverse matrix.